Recall

Start with a real vector space V.
An inner product $\langle\ ,\ \rangle$ is a f'n: $V \times V \to \mathbb{R}$ satisfying

(i) $\langle x, y\rangle = \langle y, x\rangle$, $\forall x, y \in V$ (symmetric)

(ii) $\langle x, c_1 y_1 + c_2 y_2\rangle = c_1\langle x, y_1\rangle + c_2\langle x, y_2\rangle$,
$\forall c_1, c_2 \in \mathbb{R}$ & $y_1, y_2 \in V$

<u>Note</u> (i) & (ii) $\Rightarrow \langle c_1 x_1 + c_2 x_2, y\rangle$
$= c_1\langle x_1, y\rangle + c_2\langle x_2, y\rangle$

so that $\langle\ ,\ \rangle$ is bilinear.

(iii) $x \neq 0 \Rightarrow \langle x, x\rangle > 0$ & $\langle 0, 0\rangle = 0$ (positive definite)

<u>norm on V</u>

$\|x\| > 0$ if $x \neq 0$, $\|0\| = 0$

$\|cx\| = |c|\,\|x\|$

$\|x+y\| \leq \|x\| + \|y\|$ → $\boxed{\Delta\text{-inequality}}$

inner product → norm → metric
\+ complete: Hilbert space; norm + complete: Banach space

Review of $\mathcal{L}_2$

$$\mathcal{L}_2(P) = \{[y] : E_P(y^2) < \infty\}$$

where $[y]$ denotes all equivalent rv's. We will drop $[\ ]$ & when clear set $\mathcal{L}_2 = \mathcal{L}_2(P)$.

Call $\mathcal{L}_2(P)$ V for now.

For $y_1, y_2 \in V$ define

$$\langle y_1, y_2 \rangle = E(y_1 y_2), \quad \|y_1\|^2 = \langle y_1, y_1 \rangle$$

$\langle\ ,\ \rangle$ is an inner product on V — bilinear, symmetric & positive definite while $\|\ \|$ is a norm.

Cauchy-Schwartz Inequality

$|\langle x, y\rangle| \leq \|x\| \, \|y\|$,

with $=$ iff x & y are linearly dependent.

Proof: Assume $\|x\|, \|y\| > 0$ (ow obviously true)

$$0 \leq \| \, \|y\| x + \|x\| y \|^2$$

$$= \langle \|y\| x + \|x\| y, \|y\| x + \|x\| y \rangle$$

$$= \|y\|^2 \|x\|^2 + 2 \|y\| \|x\| \langle x, y\rangle + \|x\|^2 \|y\|^2$$

$$\Rightarrow$$

$$0 \leq \|x\| \|y\| + \langle x, y\rangle$$

qed

Triangle Inequality

$$\|x+y\| \leq \|x\| + \|y\|$$

Proof.

$$\|x+y\|^2 = |\langle x+y, x+y\rangle|$$

$$= | \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 |$$

$$\leq (\|x\| + \|y\|)^2$$

qed

Parallelogram Law (// Law)

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Proof: same computation as Δ-inequality

<u>Theorem</u> $\|\cdot\|$ is a cts function on V. $\langle\ ,\ \rangle$ is cts on $V \times V$.

<u>Proof (only for $\|\ \|$)</u>

Assume $y_n \xrightarrow{\mathcal{L}_2} y$, where $y \in \mathcal{L}_2$. Then

$$\|y_n\| \leq \|y\| + \|y_n - y\| \longrightarrow \|y\|$$

$$\Rightarrow \overline{\lim}\, \|y_n\| \leq \|y\|$$

&

$$\|y\| \leq \|y_n\| + \|y - y_n\|$$

$$\Rightarrow \underline{\lim}\, \|y_n\| \geq \|y\|$$

$$\therefore \lim \|y_n\| = \|y\|$$

<u>qed</u>

<u>Theorem</u> V is a complete metric space under $\| \ \|$

(i.e. $y_n \in V$ & $\|y_n - y_m\| \to 0$ as $n, m \to \infty \Rightarrow \exists\, y \in V \ni \|y_n - y\| \to 0$)

<u>Proof</u>: Assume $y_n - y_m \xrightarrow{L_2} 0$.

Let $\epsilon > 0$. Then

$$P(|y_n - y_m| > \epsilon) \leq \frac{1}{\epsilon^2} E_p(y_n - y_m)^2$$

$$= \frac{\|y_n - y_m\|^2}{\epsilon^2} \to 0$$

$\therefore \exists$ a subseq $y_{n_k} \xrightarrow{as} y$. Now

$$E_p(y)^2 = E_p\left[\lim (y_{n_k}^2)\right] \leq \underline{\lim}\, E_p(y_{n_k}^2)$$

But $\{\|y_n\|\}$ is Cauchy & $\therefore$ bounded. $\therefore$ $\|y\| < \infty$

Now consider $\|y_n - y\|$. Let $\epsilon > 0$ & take $N > 0$ large enough so that $n, m \geq N$

$$\Rightarrow \quad \|y_n - y_m\| \leq \epsilon$$

Then

$$\|y_n - y\| = \|\lim_{k\to\infty}(y_n - y_{n_k})\|$$

$$\overset{\text{Fatou}}{\leq} \varliminf_{k\to\infty} \|y_n - y_{n_k}\|$$

$$\leq \epsilon \qquad (\text{for } n \geq N)$$

$$\therefore \quad \varlimsup \|y_n - y\| \leq \epsilon$$

$$\Rightarrow \quad \|y_n - y\| \to 0$$

QED

We will be interested in subsets W of V which are linear "manifolds". If W is closed under finite linear combinations it is called a linear manifold in V. If W is a closed set then W is called a subspace (this will be the case if W is finite dimensional). Let $B \subset V$ & let B_1 be the set all all finite linear combinations of elements of B. B_1 is the linear manifold spanned by B & its closure is the subspace spanned by B.

<u>Note</u> Our subspaces are topologically closed in the sense that they include all their limit points (this is the same as saying they include all limits of convergent sequences of elements of the subspace).

<u>Def'n</u>

Let W be a subspace of V. $\hat{y} \in W$ is an orthogonal projection of y onto W if

$$\|y - \hat{y}\| = \inf_{x \in W} \|y - x\|$$

Theorem $\exists$ a unique orthogonal projection of y onto W.

Proof: Let $d = \inf_{z \in W} \| y - z \|$.

Take $x_n \in W$ so that $\| y - x_n \| \to d$.

Apply the // Law

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

with $u = \frac{x_m - x_n}{2}$, $v = \underbrace{\frac{x_m + x_n}{2}}_{\in W} - y$,

to get

$$\left\| \frac{x_m - x_n}{2} \right\|^2 + \underbrace{\left\| \frac{x_m + x_n}{2} - y \right\|}_{\geq d}^2$$

$$= \frac{1}{2} \underbrace{\| x_m - y \|}_{\to d}^2 + \frac{1}{2} \underbrace{\| x_n - y \|}_{\to d}^2$$

$$\Rightarrow \| x_m - x_n \| \to 0$$

Hence $\{x_n\}$ is Cauchy & so $\exists$ $x \in V$ $\ni$

$$x_n \xrightarrow{L_2} x$$

But $x_n \in W$ & W is closed (so that limits of convergent sequences of W are in W). $\therefore$ $x \in W$ and so $\exists$ $x \in W$ such that

$$\|y-x\| = \inf_{z \in W} \|y-z\| \quad (*)$$

If x' also satisfies $(*)$ then

$$\underbrace{\|y-x'\|}_{d} \leq \underbrace{\|y-x\|}_{d} + \|x'-x\|$$

$\Rightarrow \|x'-x\|$ $\Rightarrow x' = x$

qed

Def'n x is orthogonal to y if $\langle x, y \rangle = 0$.

Notation $x \perp y$

Pythagorean Theorem

$$x \perp y \iff \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

First def'n of orthogonal projection

(*) $\hat{y} \in W$ is an orthogonal projection of y onto W if $\|y-\hat{y}\| = \inf_{z \in W} \|y-z\|$

2nd def'n

(**) $\hat{y} \in W$ is an orthogonal projection of y onto W if

$$(y-\hat{y}) \perp z, \ \forall z \in W$$

<u>Theorem</u> $\exists$ a unique sol'n of (**) & the 2 def'ns are equivalent. Further

$$\|y\|^2 = \|\underbrace{\hat{y}}_{\text{projection}}\|^2 + \|\underbrace{y-\hat{y}}_{\text{residual}}\|^2$$

<u>Proof</u> We first show (*) & (**) are equivalent.

($\Longrightarrow$)

Let $\hat{y}$ satisfy (*). For $z \in W$ we have $\hat{y} + s z \in W$, $\forall s \in \mathbb{R}$.

Now $\|y - (\hat{y} + s z)\|^2 \geq \|y - \hat{y}\|^2$

& so, setting $\hat{e} = y - \hat{y}$,

$$\|\hat{e}\|^2 - 2s\langle \hat{e}, z\rangle + s^2\|z\|^2 \geq \|\hat{e}\|^2$$

$$\Longrightarrow s^2\|z\|^2 - 2s\langle \hat{e}, z\rangle \geq 0, \ \forall s \in \mathbb{R}$$

$$\Longrightarrow \langle \hat{e}, z\rangle = 0 \quad \Longrightarrow (**)$$

($\Leftarrow$) Let $\hat{y}$ satisfy (**). Then for any $z \in W$

$$(y-\hat{y}) \perp (\hat{y}-z)$$

Pythagoras $$\|y-z\|^2 = \|y-\hat{y}\|^2 + \|\hat{y}-z\|^2$$

$$\Rightarrow \|y-z\|^2 \geq \|y-\hat{y}\|^2, \ \forall z \in W$$

so that (*) holds.

We have shown $\exists$ a unique sol'n to (*) and hence $\exists$ a sol'n to (**). Since every sol'n to (**) solves (*) this sol'n is unique.

Finally,

$$(y-\hat{y}) \perp \hat{y} \Rightarrow \|y\|^2 = \|y-\hat{y}\|^2 + \|\hat{y}\|^2$$

qed