More on transition functions

We assume $\{P(t), t \geq 0\}$ to be a standard stochastic semigroup unless stated otherwise. In particular we have the CKE's, $\sum_j p_{ij}(t) = 1, \forall t$ & $P(t) \to I$.

<u>Proposition</u> If $\{P(t), t \geq 0\}$ is standard then

(a) $p_{ij}(t)$ is continuous $\forall t$

(b) $p_{ii}(t) > 0$, $\forall t$

<u>Remark</u> A chain is called irreducible if $\forall i, j$ $p_{ij}(t) > 0$ for some $t > 0$. It turns out that either $p_{ij}(t) = 0, \forall t > 0$ or $p_{ij}(t) > 0$, $\forall t > 0$. A pf $\underset{\sim}{\pi}$ is a stationary dist'n of a chain if $\underset{\sim}{\pi}' = \underset{\sim}{\pi}' P(t)$, $\forall t \geq 0$. If $\{P(t)\}$ is standard then either a stationary dist'n exists & is unique in which case $p_{ij}(t) \to \pi_j$ or there is none in which case $p_{ij}(t) \to 0$. It turns out that under conditions $\underset{\sim}{\pi}' = \underset{\sim}{\pi}' P(t)$, $\forall t \geq 0$ iff $\underset{\sim}{\pi}' Q = \underset{\sim}{0}'$.

Proof: For (a)

Use the Chapman-Kolmogorov equations to obtain

$$|p_{ij}(t+h) - p_{ij}(t)| = \left|\sum_k (p_{ik}(h) - \delta_{ik})\, p_{kj}(t)\right|$$
$$\le (1 - p_{ii}(h))\, p_{ij}(t) + \sum_{k \ne i} p_{ik}(h)$$
$$\le (1 - p_{ii}(h)) + (1 - p_{ii}(h)) \to 0$$

For (b)

Since $p_{ii}(t) \to 1$ as $t \downarrow 0$ we have $p_{ii}(t) > 0$ $\forall\, t \in [0, h]$, for some $h > 0$. Now for any t choose n large enough so that $t \le hn$. Now use the CKE to get

$$P(t) = P\left(\tfrac{t}{n} + \cdots + \tfrac{t}{n}\right) = \left[P\left(\tfrac{t}{n}\right)\right]^n$$

so that
$$p_{ii}(t) \ge (p_{ii}(t/n))^n > 0$$

qed

<u>Lemma</u> Suppose $g : (0, \infty) \to [0, \infty)$ satisfies

(i) $g(s+t) \le g(s) + g(t)$, $\forall s, t > 0$

(ii) $\lim_{t \downarrow 0} g(t) = 0$ (set $g(0) = 0$ for notational purposes)

Then
$$\bar{g} = \lim_{t \downarrow 0} \frac{g(t)}{t} = \sup_{t > 0} \frac{g(t)}{t}$$

<u>Note</u> $\bar{g}$ may be $+\infty$

<u>Proof</u> Let $t, h > 0$ & set $m = \left[\frac{t}{h}\right]$ to be the greatest integer $\le \frac{t}{h}$ (= integer part of $\frac{t}{h}$ in this case). Then

$$t = mh + r, \quad \text{where } 0 \le r < h$$

So $g(t) = g(mh+r) \le g(mh) + g(r) \le m g(h) + g(r)$

$$\therefore \frac{g(t)}{t} \le \frac{m g(h)}{t} + \frac{g(r)}{t}$$
$$= \frac{g(h)}{h}\left(\frac{mh}{t}\right) + \frac{g(r)}{t}$$

This yields
$$\varliminf_{h \to 0^+} \frac{g(t)}{t} \le \varliminf \left[\frac{g(h)}{h} \underset{\substack{\downarrow \\ 1}}{\frac{mh}{t}} + \underset{\substack{\downarrow \\ 0}}{\frac{g(r)}{t}} \right]$$

$$\Rightarrow \frac{g(t)}{t} \le \varliminf \frac{g(h)}{h}$$

$$\Rightarrow \sup_{t>0} \frac{g(t)}{t} \leq \varliminf_{h\to 0^+} \frac{g(h)}{h}$$

Now $$\varlimsup_{h\downarrow 0} \frac{g(h)}{h} \leq \varlimsup_{h\downarrow 0} \sup_{t\geq h} \frac{g(t)}{t}$$

$$= \lim_{h\downarrow 0} \sup_{t\geq h} \frac{g(t)}{t} \leq \sup_{t>0} \frac{g(t)}{t}$$

$$\therefore \varlimsup_{h\downarrow 0} \frac{g(h)}{h} \leq \sup_{t>0} \frac{g(t)}{t} \leq \varliminf \frac{g(h)}{h}$$

$$\Rightarrow \varlimsup \frac{g(h)}{h} = \varliminf \frac{g(h)}{h} = \sup_{t>0} \frac{g(t)}{t}$$

qed

<u>Proposition</u> Let $\{P(t), t \geq 0\}$ be standard. Then

(i) $\dot{p}_{ii}(0)$ exists (may be $-\infty$) — set $q_i = -\dot{p}_{ii}(0)$

(ii) $p_{ii}(t) \geq e^{-q_i t}$

(iii) $q_i = 0 \Leftrightarrow p_{ii}(t) = 1, \; \forall t \geq 0$

(iv) $\sum_{j \neq i} q_{ij} \leq q_i$, where $q_{ij} = \dot{p}_{ij}(0)$

exists (& is finite)

<u>Remark</u> We won't prove $\dot{p}_{ij}(0)$ exists here. Also, under general conditions $\sum_{j \neq i} q_{ij} = q_i$.

<u>Proof</u> (i) We first note that $p_{ii}(t) > 0, \; \forall t \geq 0$.
Now set $g(t) = -\log(p_{ii}(t))$. By the CKE

$$p_{ii}(t+s) \geq p_{ii}(t)\, p_{ii}(s) \quad , \; \forall s, t \geq 0$$

& so

$$g(t+s) \leq g(t) + g(s)$$

$$\Rightarrow q_i = \lim_{t \downarrow 0} \frac{g(t)}{t} \quad \text{exists (may be } +\infty)$$

Now

$$-\dot{p}_{ii}(0) = \lim_{t\downarrow 0} \frac{1-p_{ii}(t)}{t}$$

$$= \lim_{t\downarrow 0} \underbrace{\left[\frac{1-e^{-g(t)}}{g(t)}\right]}_{\to 1} \frac{g(t)}{t}$$

$$= \lim_{t\downarrow 0} \frac{g(t)}{t} = g_i$$

which shows (i).

(ii) $g_i = \sup_{t>0} \frac{g(t)}{t} \Rightarrow g_i \geq \frac{g(t)}{t}$

$$\Rightarrow g(t) \leq g_i t$$

$$\Rightarrow p_{ii}(t) \geq e^{-g_i t}$$

(iii) $g_i = \sup_{t>0} \frac{g(t)}{t}$ & so $g_i = 0 \Rightarrow g(t) = 0, \forall t > 0$

$$\Rightarrow p_{ii}(t) = 1, \forall t > 0$$

$$\Rightarrow p_{ii}(t) = 1, \forall t \geq 0$$

On the other hand $p_{ii}(t) = 1, \forall t \geq 0 \Rightarrow g(t) = 0$ for $t > 0 \Rightarrow g_i = 0$

(iv) Let A be a finite set of states. Then

$$\sum_{j\neq i, j\in A} p_{ij}(t) \le \sum_{j\neq i} p_{ij}(t) = 1 - p_{ii}(t)$$

$$\Rightarrow \sum_{\substack{j\neq i \\ j\in A}} \frac{p_{ij}(t)}{t} \le \frac{1-p_{ii}(t)}{t}$$

$$\overset{t\downarrow 0}{\Longrightarrow} \sum_{\substack{j\neq i \\ j\in A}} q_{ij} \le q_i \Rightarrow \sum_{j\neq i} q_{ij} \le q_i$$

qed

<u>Proposition</u> Let $\{P(t)\}$ be standard with Q-matrix Q. We have

(1) If for <u>some</u> state i $\sum_{j\neq i} q_{ij} = q_i < \infty$

then $\dot{p}_{ij}(t)$ exists $\forall j$ &

$$\dot{p}_{ij}(t) = \sum_k q_{ik}\, p_{kj}(t), \quad \forall t \geq 0, \forall j \quad (BE)$$

(2) If $q_i \le M < \infty$ $\forall i$ then

$$\dot{p}_{ij}(t) = \sum_k p_{ik}(t)\, q_{kj}, \quad \forall t \geq 0, i, j \quad (FE)$$

<u>Remark</u> If $\sup_i q_i < \infty$ then $\{P_t\}$ is <u>uniform</u>

See end of lecture.

Proof

For (1) $$p_{ij}(t+\Delta) - p_{ij}(t) = \left(\sum_k p_{ik}(\Delta) p_{kj}(t)\right) - p_{ij}(t)$$

$$= \sum_{k \neq i} p_{ik}(\Delta) p_{kj}(t) + (p_{ii}(\Delta) - 1) p_{ij}(t)$$

Now let A be a finite set of states. We then have

$$\lim_{\Delta \downarrow 0} \sum_{k \neq i} \frac{p_{ik}(\Delta) p_{kj}(t)}{\Delta} \geq \lim_{\Delta \downarrow 0} \sum_{\substack{k \neq i \\ k \in A}} \frac{p_{ik}(\Delta)}{\Delta} p_{kj}(t)$$

$$= \sum_{\substack{k \neq i \\ k \in A}} q_{ik} p_{kj}(t)$$

Now let $A \uparrow$ state space to get

$$\underline{\lim} \sum_{k \neq i} \frac{p_{ik}(\Delta)}{\Delta} p_{kj}(t) \geq \sum_{k \neq i} q_{ik} p_{kj}(t)$$

We also have

$$\overline{\lim} \sum_{k \neq i} \frac{p_{ik}(\Delta)}{\Delta} p_{kj}(t) = \overline{\lim} \left[\sum_{\substack{k \neq i \\ k \in A}} \frac{p_{ik}(\Delta) p_{kj}(t)}{\Delta} \right.$$

$$\left. + \sum_{\substack{k \neq i \\ k \in A^c}} \frac{p_{ik}(\Delta) p_{kj}(t)}{\Delta} \right]$$

$$\leq \overline{\lim} \left[\sum_{\substack{k \neq i \\ k \in A}} \frac{p_{ik}(\Delta)}{\Delta} p_{kj}(t) + \sum_{\substack{k \neq i \\ k \in A^c}} \frac{p_{ik}(\Delta)}{\Delta} \right]$$

Now,

$$\sum_{\substack{k\neq i\\ k\in A^c}} \frac{p_{ik}(\Delta)}{\Delta} + \sum_{\substack{k\neq i\\ k\in A}} \frac{p_{ik}(\Delta)}{\Delta} = \frac{1-p_{ii}(\Delta)}{\Delta} \to q_i \text{ as } \Delta\downarrow 0$$

Hence

$$\overline{\lim} \sum_{k\neq i} \frac{p_{ik}(\Delta)}{\Delta} p_{kj}(t)$$

$$\leq \overline{\lim}\left[\sum_{\substack{k\neq i\\ k\in A}} \frac{p_{ik}(\Delta)}{\Delta} p_{kj}(t) - \sum_{\substack{k\neq i\\ k\in A}} \frac{p_{ik}(\Delta)}{\Delta} + \frac{1-p_{ii}(\Delta)}{\Delta}\right]$$

$$= \lim_{\Delta\downarrow 0}\left[\sum_{\substack{k\neq i\\ k\in A}} \frac{p_{ik}(\Delta)}{\Delta} p_{kj}(t) - \sum_{\substack{k\neq i\\ k\in A}} \frac{p_{ik}(\Delta)}{\Delta} + \frac{1-p_{ii}(\Delta)}{\Delta}\right]$$

$$= \sum_{\substack{k\neq i\\ k\in A}} q_{ik} p_{kj}(t) - \sum_{\substack{k\neq i\\ k\in A}} q_{ik} + q_i$$

Now let $A \uparrow$ state space & note

$$q_i - \sum_{\substack{k\neq i\\ k\in A}} q_{ik} \to 0$$

to get $\overline{\lim} \sum_{k\neq i} \frac{p_{ik}(\Delta)}{\Delta} p_{kj}(t) = \sum_{k\neq i} q_{ik} p_{kj}(t)$

& so $\lim_{\Delta\downarrow 0} \sum_{k\neq i} \frac{p_{ik}(\Delta)}{\Delta} p_{kj}(t) = \sum_{k\neq i} q_{ik} p_{kj}(t)$

Now let $s \downarrow 0$ in our first eq'n to get (after dividing by s)

$$\dot{p}_{ij}(t) = \left(\sum_{k \neq i} g_{ik} p_{kj}(t)\right) - g_i\, p_{ij}(t)$$

That is,

$$\dot{p}_{ij}(t) = \sum_{k} g_{ik}\, p_{kj}(t) \quad , \forall j \;\&\; t \geq 0$$

For (2)

$$\frac{p_{ij}(t+s) - p_{ij}(t)}{s} = \sum_{k} p_{ik}(t)\left(\frac{p_{kj}(s) - \delta_{kj}}{s}\right),$$

where $\delta_{kj} = 1$ if $k = j$ & 0 otherwise. Since

$$\left|\frac{p_{kj}(s) - \delta_{kj}}{s}\right| \leq \frac{1 - p_{kk}(s)}{s} \leq \frac{1 - e^{-g_k s}}{s} \leq g_k \leq M$$

we can use the DCT to get

$$\dot{p}_{ij}(t) = \sum_{k} p_{ik}(t)\, g_{kj} \quad , \forall i,j \;\&\; t \geq 0$$

qed

Def'n A square matrix Q is called a Q-matrix if $0 \le q_{ij} < \infty$ for $i \ne j$, $-\infty \le q_{ii} \le 0$ & $\sum_j q_{ij} \le 0$ $(\forall i)$.

It is stable if $q_i = -q_{ii} < \infty$ $\forall i$. It is conservative if $q_i < \infty$, $\forall i$ and $\sum_j q_{ij} = 0$, $\forall i$.

So Q conservative $\Rightarrow$ BE $\forall i, j, t \ge 0$

Q conservative + $\sup_i q_i < \infty \Rightarrow$ FE $\forall i, j, t \ge 0$

These conditions can be relaxed.

Now take a conservative Q & look at the BE

$$\dot{p}_{ij}(t) = \sum_k q_{ik} p_{kj}(t)$$

which is

$$\dot{p}_{ij}(t) + q_i p_{ij}(t) = \sum_{k \ne i} q_{ik} p_{kj}(t)$$

Multiply by the "integrating factor" $e^{q_i t}$ to get

$$\frac{d}{dt}\left[e^{q_i t} p_{ij}(t)\right] = e^{q_i t} \sum_{k \ne i} q_{ik} p_{kj}(t)$$

This leads to the integral form of the BE

$$p_{ij}(t) = \delta_{ij} e^{-q_i t} + \int_0^t e^{-q_i(t-s)} \sum_{k \neq i} q_{ik} p_{kj}(s) ds$$

Now define

$$p_{ij}^{(0)}(t) = \delta_{ij} e^{-q_i t}$$

$$p_{ij}^{(n+1)}(t) = \delta_{ij} e^{-q_i t} + \int_0^t e^{-q_i(t-s)} \sum_{k \neq i} q_{ik} p_{kj}^{(n)}(s) ds, \quad n \geq 0$$

Then

$$0 \leq p_{ij}^{(n)}(t) \leq p_{ij}^{(n+1)}(t) \leq 1$$

so that

$$f_{ij}(t) = \lim_{n \to \infty} p_{ij}^{(n)}(t)$$

exists. In fact $0 \leq f_{ij}(t) \leq 1$, $f_{ij}(t)$ satisfies the BE + CKE, $f_{ij}(t) \to \delta_{ij}$ as $t \downarrow 0$ and $\sum_j f_{ij}(t) \leq 1$ (improper possibilities). If $p_{ij}(t)$ is any solution of the BE then

$$p_{ij}(t) \geq \delta_{ij} e^{-q_i t} = p_{ij}^{(0)}(t)$$

so that by induction $p_{ij}(t) \geq p_{ij}^{(n)}(t)$ & hence

$$p_{ij}(t) \geq f_{ij}(t)$$

For this reason we refer to $f_{ij}(t)$ as the minimal solution of the BE. It turns out that in the case where $f_{ij}(t)$ is a proper dist'n ($\sum_j f_{ij}(t) = 1$) it is the only solution.

If we consider the uniform semigroup case we obtain a clearer picture.

<u>Theorem</u> If $\{P(t), t \geq 0\}$ is a uniform semigroup with generator Q then it is the unique solution to both

$$\dot{P}(t) = P(t)\,Q \qquad \text{(FE)}$$

$$\dot{P}(t) = Q\,P(t) \qquad \text{(BE)}$$

(with boundary condition $P(0) = I$). Furthermore $Q\underset{\sim}{1} = \underset{\sim}{0}$ (ie $\sum_j q_{ij} = 0, \forall i$) & $P(t) = e^{tQ}$.

<u>Standard stochastic semigroup</u>

$P(0) = I$, $P(s+t) = P(s)\,P(t)$, $\forall s,t \geq 0$, $P(t)\,\underset{\sim}{1} = \underset{\sim}{1}$, $P(t) \to P(0)$

<u>Uniform stochastic semigroup</u>

Standard + $P(t) \to P(0)$ uniformly

<u>Remark</u> $p_{ii}(t) \to p_{ii}(0) = 1$ uniformly in i <u>implies</u>

$$\sup_{i \neq j} p_{ij}(t) \leq \sup_i \left[1 - p_{ii}(t)\right] \to 0$$

<u>Theorem</u> $\{P(t)\}$ is uniform iff $\sup_i q_i < \infty$

<u>Theorem</u> $\{P(t)\}$ uniform with generator Q then it is the unique sol'n of

$$\dot{P}(t) = P(t)\,Q \quad (FE) \quad + \quad \dot{P}(t) = Q\,P(t) \quad (BE)$$

subject to $P(0) = I$. In addition

$$P(t) = e^{tQ} \quad \text{and} \quad Q\,\underset{\sim}{1} = \underset{\sim}{0}$$

<u>Remark</u> ① Let $A = \{a_{ij}\}_{i \in S, j \in S}$ (state space) be such that

$$\sup_i \sum_j |a_{ij}| < \infty$$

Such a matrix can be the generator of a uniform semigroup iff $a_{ij} \geq 0$ for $i \neq j$ + $A\,\underset{\sim}{1} = \underset{\sim}{0}$.

② For a birth process the uniform condition is $\sup_i \lambda_i < \infty$ which is sufficient for both FE + BE to have unique sol'ns. In fact the weaker $\sum_i \frac{1}{\lambda_i} = \infty$ is necessary + sufficient.