Branching processes

Let $Z_{i1}, Z_{i2}, \ldots$ $i = 0, 1, 2, \ldots$ be iid counting rv's with pgf G.
Define the stochastic process $\{X_0, X_1, \ldots\}$ by $X_0 = 1$ and

$$X_n = Z_{n-1,1} + \cdots + Z_{n-1, X_{n-1}}$$

The process $\{X_t \mid t = 0, 1, \ldots\}$ is called a Galton Watson branching process with offspring pgf G.
Denote the offspring mean & variance by μ & σ^2 respectively ($\mu = E(Z_{i1})$, $\sigma^2 = Var(Z_{i1})$).
Branching processes are classified as subcritical, critical or supercritical as $\mu < 1$, $\mu = 1$ or $\mu > 1$.

Remarks

- clearly $X_t \mid \underset{\sim}{X}_{t-1} \overset{d}{=} X_t \mid X_{t-1}$ so that the process is Markov. Also, $X_{t+s} \mid X_t \overset{d}{=} X_s \mid X_0$ so that the process is time homogeneous.

- the assumption $X_0 = 1$ is for convenience & is easily dropped

- X_t is the population at the t-th generation. Individuals live for one generation and have offspring according to the offspring dist'n.

Set $G_t(s) = E(s^{X_t})$, $p_t = P(X_t = 0)$,
$p = \lim_{t \to \infty} p_t$ & $G_{(t)}(s) = t$-th iterate of G

We then note that p_t increases and since it is bounded the limit p exists. It is called the probability of ultimate extinction. We also have

$$G_n(s) = G[G_{n-1}(s)]$$

which incidentally yields

$$G_n(s) = G_{(n)}(s)$$

So $p_n = P(X_n = 0) = G_n(0) = G[G_{n-1}(0)] = G(p_{n-1})$

and hence

$$p = G(p)$$

Now notice $G'(s)$ and $G''(s) > 0$ on $s > 0$ so that G is strictly increasing & convex on $s > 0$.

If $\mu = G'(1) \leq 1$ there will only be one root of

$$\rho = G(\rho) \quad ; \quad 0 \leq \rho \leq 1$$

and that is at $\rho = 1$ (certain extinction).

If $\mu > 1$ then there will be 2 roots - one at $\rho = 1$ and the other at $\rho < 1$. It is clear that ρ must be the smaller of the 2 roots. ~~Indeed~~, call the smaller root ρ_0. Then we have

$$\rho_1 \leq \rho_0 \Rightarrow \rho_2 = G(\rho_1) \leq G(\rho_0) = \rho_0 \Rightarrow \dots \rho_n \leq \rho_0$$

so that $\lim \rho_n \leq \rho_0$.

Notice that $E(X_t) = \mu^t$. If

the process does not die off ($\mu > 1$ case <u>and</u> condition on non-extinction) then it can be shown $X_n \xrightarrow{as} \infty$. Set

$$W_n = \frac{X_n}{E(X_n)}$$

so that $E(W_n) = 1$. Use pgf to show

$$Var(W_n) = \frac{\sigma^2 (1 - \mu^{-n})}{\mu^2 - \mu}$$

so that

$$Var(W_n) \to \frac{\sigma^2}{\mu^2 - \mu} \neq 0$$

In fact we can show $W_n \xrightarrow{as} W$ using martingale arguments. We return to this matter latter.

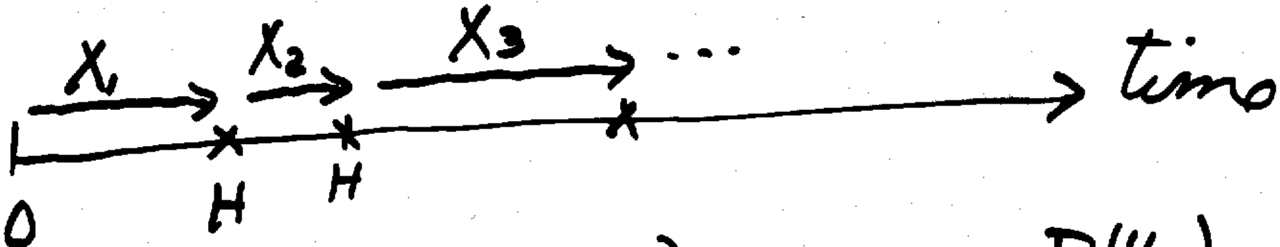
A bit on recurrence / renewal

Let $X_1, X_2, \cdots$ be independent rv's $\in \mathbb{N} = \{1, 2, \cdots\}$. Furthermore take $X_2, X_3, \cdots$ to be iid with pgf G and set $D(s) = E(s^{X_1})$.

At times

$$T_k = X_1 + \cdots + X_k \quad , \; k = 1, 2, \cdots$$

an "event" H occurs. At other times it does not. The X's are called <u>interarrival</u>/interoccurrence times.



Let $H_n = \{H \text{ occurs at time } n\}$ & $u_n = P(H_n)$.

Let

$$U(s) = \sum_{n \geq 1}^{\infty} u_n s^n$$

be the generating function of $\{u_1, u_2, \cdots\}$. This is not a pgf! We can show

$$U(s) = \frac{D(s)}{1 - G(s)}$$

which allows us to calculate u_n, at least in principle. Consider, for example, set

$$D_*(s) = \frac{1 - G(s)}{\mu\,(1-s)}, \quad |s| < 1$$

where $\mu = E(X_2)$ is the mean interarrival time. Let

$$D_*(1) = \lim_{s \uparrow 1} D_*(s)$$

$$= \frac{-G'(1)}{-\mu} = 1$$

Now expand $D_*(s)$ in a power series and notice that the coefficient of s^n is

$$d_n \equiv \mu^{-1}\left[1-\left(P(X_2=1)+\cdots+P(X_2=n)\right)\right]$$

$$= P(X_2>n)/\mu \geq 0 .$$

Since

$$E(X_2)=\sum_{n=0}^{\infty} P(X_2>n)$$

we conclude

$$D_*(s) = \sum_{n=0}^{\infty} d_n s^n$$

is a pgf (since $d_n \geq 0$ & $\sum d_n = 1$).

So in this case

$$U(s) = \frac{1}{\mu(1-s)}$$

which yields $u_n = \frac{1}{\mu}$; $n=1,2,\cdots$

which is a <u>constant</u> sequence.

If we take $X_1 \sim D_*$ — notice that $X_1 \in \mathbb{N} \cup \{0\} = \mathbb{Z}^+$ then

$$u_n = \frac{1}{\mu} \rightarrow \frac{1}{\mu}$$

This suggests $u_n \rightarrow 1/\mu$ more generally since the initial distribution's influence will fade in time. This result is the renewal theorem.

<u>Renewal Theorem</u>

If μ is finite and $\gcd\{m : P(X_2 = m) > 0\} = 1$ then $u_n \rightarrow 1/\mu$

This is an important result and may be proved in a variety of ways. A modern approach is to use "<u>coupling</u>". For now we show

$$U(s) = D(s) / [1 - G(s)]$$

Suppose $X_1, X_2, \cdots$ are iid & $\in \{1, 2, \cdots\}$
Let the common pgf be

$$G(s) = E(s^{X_1}) = E(s^{X_2}) = \cdots$$

Now let $u_n = P(\text{renewal at time } n)$ &

$$U(s) = \sum_{n=1}^{\infty} u_n s^n,$$

which is the generating function of $u_1, u_2, \cdots$. Now let T_n be the time of the nth renewal.
So

$$T_n = X_1 + \cdots + X_n$$

Look at $H_n = \{\text{renewal at time } n\}$ & its indicator rv $I(H_n)$.

$$I(H_n) = 1 \quad , \text{if a renewal at time } n$$

$$= 0 \quad , \text{ o w}$$

Clearly $E[I(H_n)] = u_n$

Now

$$\sum_{n=1}^{\infty} I(H_n) s^n$$

converges at least on $|s|<1$, +

$$E\left[\sum_{n=1}^{\infty} I(H_n) s^n\right] = \sum_{n=1}^{\infty} E[I(H_n)] s^n$$

$$= \sum_{n=1}^{\infty} u_n s^n = U(s)$$

Now look at

$$\sum_{n=1}^{\infty} s^{T_n}$$

which also converges for $|s|<1$, +

$$E\left(\sum_{n=1}^{\infty} s^{T_n}\right) = \sum_{n=1}^{\infty} E\left(s^{T_n}\right)$$

$$= \sum_{n=1}^{\infty} E\left(s^{X_1+X_2+\cdots+X_n}\right)$$

$$= \sum_{n=1}^{\infty} E(s^{X_1}) E(s^{X_2}) \cdots E(s^{X_n})$$

$$= \sum_{n=1}^{\infty} G(s)^n$$

$$= \frac{G(s)}{1-G(s)} \quad \left(= \frac{D(s)}{1-G(s)} \text{ if } D(s) = E(s^{X_1}) \right)$$

But

$$\sum_{n=1}^{\infty} I(H_n) s^n = \sum_{n=1}^{\infty} s^{T_n}$$

[this is always true even if X_1 has a different distribution than $X_2, X_3, \cdots$]

$$\therefore \quad U(s) = \frac{G(s)}{1-G(s)}$$

The Renewal Theorem

Let $X_1 \in \{0, 1, 2, \cdots\}$ be independent of $X_2, X_3, \cdots$ which are iid nonperiodic counting rv's > 0 with pgf G. Then

$$u_n \to \frac{1}{\mu}$$

Proof: Define a $*$-renewal process such that $X_1^*, X_2^*, \cdots$ are ind of the X's; X_1^* has pgf D_* & $X_2^*, X_3^*, \cdots$ are iid with pgf G. We know that

$$u_n^* \to \frac{1}{\mu}$$

Now let T be the first time where H occurs (simultaneously) in both processes. "Clearly" $T \overset{wp1}{<} \infty$. After T the $*$ process & the original process are indistinguishable. So

$$u_m = P(H_m \mid T \le m)\, P(T \le m) + P(H_m \mid T > m)\, P(T > m)$$

$$= P(H_m^* \mid T \le m)\, P(T \le m) + P(H_m \mid T > m)\, P(T > m)$$

{ H_m is the event that H occurs at time m for the original process & H_m^* refers to the * process }

Also

$$\frac{1}{\mu} = u_m^* = P(H_m^* \mid T \le m)\, P(T \le m) + P(H_m^* \mid T > m)\, P(T > m)$$

$$\therefore \quad |u_m - u_m^*| = P(T > m)\, |P(H_m \mid T > m) - P(H_m^* \mid T > m)|$$

$$\le 2\, P(T > m) \to 0 \text{ as } m \to \infty.$$

$$\therefore \quad u_m \to \frac{1}{\mu}$$

qed

Renewal processes

Let $X_1, X_2, \ldots$ be iid ≥ 0 rv's. Set

$$S_n = \sum_{i=1}^{n} X_i \quad , \; n \geq 1$$

and $S_0 = 0$. The S_n, $n \geq 1$, will be the times of renewals (or events) and if $N(t) = \#$ of renewals in $(0, t]$ then $\{N(t), t \geq 0\}$ will be termed a renewal (counting) process. By convention we will not start with a renewal ($N(0) = 0$). Also it will be convenient to allow X_1 to have a different distribution than X_i, $i \geq 2$. In this case we have a delayed renewal process. $\mu = E(X_n)$ is the <u>mean interarrival time</u> (the X's are called interarrival times). We will assume $P(X_n = 0) < 1$ so that $\mu > 0$.

The SLLN gives

$$\frac{S_n}{n} \xrightarrow{as} \mu$$

so that $S_n \xrightarrow{as} \infty$. This tells us that $N(t)$ is finite (wp1, but we will drop this qualification when there can be no confusion — the same approach was taken with $E(Y|X)$).

$$m(t) = E[N(t)]$$

is called the renewal function.

Incidentally, we have already seen a renewal process, namely the Poisson process. There the X's were iid exponentials and $m(t) = \lambda t$, where λ was the rate.

Clearly

$$N(t) \geq n \iff S_n \leq t$$

so that

$$P(N(t) \geq n) = P(S_n \leq t)$$

S_n is a sum of iid rv's so that its df, F_n say, is the n-fold convolution of F (the df of the X's) with itself.

<u>Remark</u> If X has df F and Y has df G & X, Y are independent then the distribution of $X+Y$ is a convolution. In terms of df's it is denoted by $F*G$ so that

$$F*G(a) = P(X+Y \leq a)$$

$$\left(= \underbrace{E_G(F(a-Y))} \text{ or } \underbrace{\int_{-\infty}^{\infty} F(a-y)\,dG}\right)$$

means same

Notice

$$P[N(t)=n] = P(S_n \le t) - P(S_{n+1} \le t)$$

Denote the df of S_n by F_n $(= F*F*\cdots*F)$.

<u>Proposition</u>

$$m(t) = \sum_{n=1}^{\infty} F_n(t) = \sum_{n=1}^{\infty} P(S_n \le t)$$

<u>Proof</u> $N(t) = \sum_{n=1}^{\infty} I_{\{0 \le S_n \le t\}}$

$$\Rightarrow E(N(t)) = \sum_{n=1}^{\infty} E\left[I_{\{0 \le S_n \le t\}}\right]$$

$$= \sum_{n=1}^{\infty} P(S_n \le t)$$

qed

<u>Proposition</u> $m(t) < \infty$, $t \geq 0$

<u>Proof</u> Since $P(X_n > 0)$ $\exists$ α st $P(X_n \geq \alpha) > 0$. Now consider the new renewal process $N_\alpha(t)$ with interarrivals

$$X_{\alpha,n} = 0 \quad , \text{ if } X_n < \alpha$$
$$= \alpha \quad , \text{ if } X_n \geq \alpha$$

This new renewal process can have renewals only at times $n\alpha$. The # of renewals at each of these times can be > 1 (since the X_α's can be 0). In fact the numbers are independent geometric$(P(X_n \geq \alpha))$. Hence

$$E[N_\alpha(k\alpha)] = k / P(X_n \geq \alpha)$$

(again $N_\alpha(0) \equiv 0$)

so that for $t > 0$

$$E[N_a(t)] \leq \frac{t/a}{P(X_n \geq a)}$$

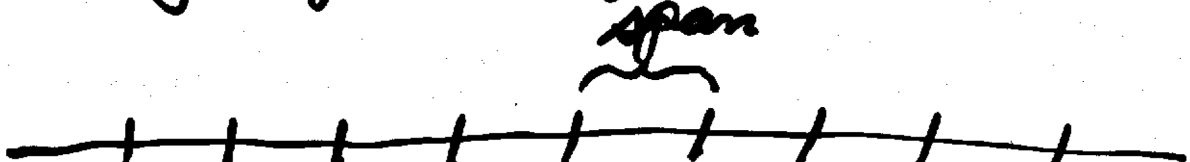
Now since $X_{a,n} \leq X_n$ we have $N_a(t) \geq N(t)$ so that $E(N(t)) < \infty$.

qed

Remarks.

(a) The same argument in fact shows $E(N^r(t)) < \infty$, for any $r > 0$!

(b) The $X_{a,n}$ are rv's that can take on values on an equally spaced grid



These are lattice rv's with the span being the largest spacing possible.

The span of a lattice rv is also called the period. For counting rv's (values $\in \{0, 1, 2, \cdots\}$) we often use the terms aperiodic or non-arithmetic if the span is 1. This can lead to some confusion since lattice rv's are also called arithmetic rv's! For integer valued rv's we will try to use the term aperiodic to mean the span is 1 and periodic for spans >1. Unfortunately the terms non-arithmetic and arithmetic may creep in.

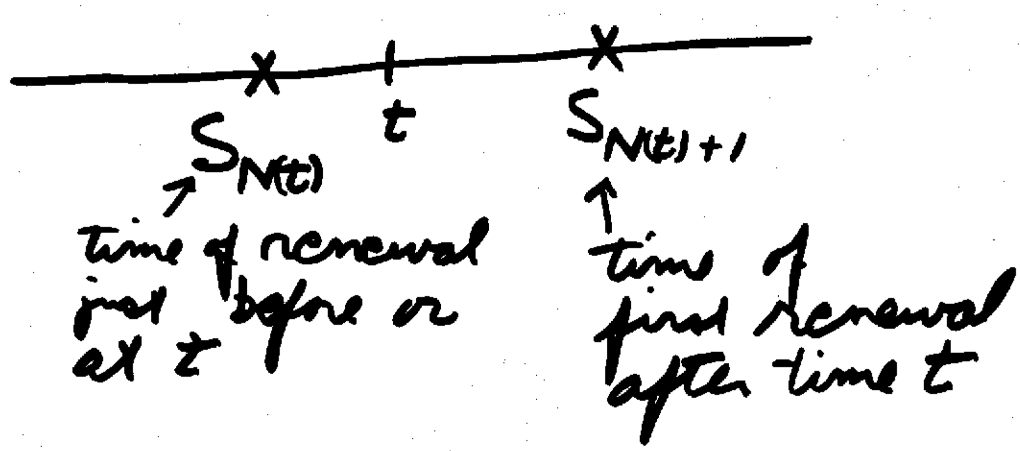
Set $N(\infty) = \lim_{t \to \infty} N(t)$. Then

$$P(N(\infty) < \infty) = P(X_n = \infty \text{ for some } n)$$
$$= P\left(\bigcup_{n=1}^{\infty} \{X_n = \infty\}\right)$$
$$\leq \sum_{n=1}^{\infty} P(X_n = \infty) = 0$$

and so $N(\infty) \overset{a.s.}{=} \infty$.

<u>Proposition</u> $\frac{N(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu}$

<u>Proof:</u> $S_{N(t)} \leq t < S_{N(t)+1}$



Hence

$$\frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} \le \frac{S_{N(t)+1}}{N(t)}$$

Since $\frac{1}{N(t)} \xrightarrow{as} 0$ we conclude

$$\frac{N(t)+1}{N(t)} \xrightarrow{as} 1$$

and

$$\frac{S_{N(t)+1}}{N(t)} = \left(\frac{S_{N(t)+1}}{N(t)+1}\right) \frac{N(t)+1}{N(t)} \xrightarrow{as} \mu$$

As $\frac{S_{N(t)}}{N(t)} \xrightarrow{as} \mu$ we obtain

$$\frac{t}{N(t)} \xrightarrow{as} \mu$$

or

$$\frac{N(t)}{t} \xrightarrow{as} \frac{1}{\mu}$$

qed

So we have $\frac{N(t)}{t} \xrightarrow{as} \frac{1}{\mu}$

but this does not imply

$$(*) \qquad \frac{E(N(t))}{t} \to \frac{1}{\mu} .$$

However, in this case (*) is true which is a result known as the Elementary Renewal Theorem. There are other versions such as

<u>Blackwell's Theorem</u>

(i) If the X's are lattice rv's with period (ie span) d then

$$E(\#\text{ of renewals at } nd) \to d/\mu$$

(ii) If the X's are not lattice rv's then for $a \geq 0$

$$m(t+a) - m(t) \to a/\mu$$

<u>Note</u> (i) holds for $d=1$

Notice that if we do not allow multiple renewals ($X_n > 0$) then

$$E(\# \text{ of renewals at } nd)$$

$$= P(\text{renewal at } nd)$$

In the case $d=1$ this is just our u_n which we proved $\to \frac{1}{\mu}$ using a coupling argument (there is still the issue of proving $T<\infty$).

If the X's are non-lattice rv's and we assume

$$\lim_{t\to\infty} m(t+a)-m(t) = g(a)$$

exists then clearly

$$g(a+b) = g(a) + g(b)$$

so that $g(a) = ca$. Now use the Elementary Renewal Theorem to conclude $c = 1/\mu$.

We will not prove Blackwell's Theorem in the non-lattice case.

A CLT for renewal processes goes along the lines

$$\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} \xrightarrow{d} N(0,1)$$

Here $\sigma^2 = \mathrm{Var}(X_n)$.

We now proceed to prove the Elementary Renewal Theorem.

<u>Def'n</u> Let $X_1, X_2, \cdots$ be a sequence of rv's. A counting rv $N > 0$ is said to be a stopping time for the X's if $\{N \overset{(=)}{\leq} n\}$ is a function of $X_1, \cdots, X_n$ for each $n = 1, 2, \cdots$.

eg Let $X_1, \dots$ be iid Bernoulli$(\frac{1}{2})$. Set $N = \min\{n : X_1 + \dots + X_n = 10\}$. Then N is a stopping time. In fact N is our familiar negative binomial.

<u>Wald's Equation</u>

Let $X_1, X_2, \dots$ be iid with mean μ and let N be a stopping time for the X's such that $E(N) < \infty$. Then

$$E\left(\sum_{n=1}^{N} X_n\right) = E(N)\,\mu$$

<u>Proof</u>

$$\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} X_n I_{\{N \geq n\}}$$

$$\Rightarrow E\left(\sum_{n=1}^{N} X_n\right) = \sum_{n=1}^{\infty} E\left(X_n I_{\{N \geq n\}}\right)$$

$$= \sum_{m=1}^{\infty} E(X_m) E\left(I_{\{N \geq m\}}\right), \quad \therefore \{N \geq m\} = \{N \leq m-1\}^c$$

$$= \mu \sum_{m=1}^{\infty} E\left(I_{\{N \geq m\}}\right) = \mu \sum_{m=1}^{\infty} P(N \geq m)$$

$$= \mu \, E(N)$$

qed

<u>Corollary</u> $E(S_{N(t)+1}) = \mu (m(t)+1)$

<u>Proof</u>: $N(t)+1 = m \iff N(t) = m-1$

$\iff X_1 + \cdots + X_{m-1} \leq t$

and $X_1 + \cdots + X_m > t$

so that $\{N(t)+1 = m\}$ depends only on $X_1, \ldots, X_m$ & is therefore independent of $X_{m+1}, X_{m+2}, \ldots$. It follows that $N(t)+1$ is a stopping time so that

$$E(S_{N(t)+1}) = \mu \, E(N(t)+1)$$

$$= \mu (m(t)+1)$$

Elementary Renewal Theorem

$$\frac{m(t)}{t} \to \frac{1}{\mu}$$

(if $\mu = \infty$ then $m(t)/t \to 0$)

Proof Suppose $\mu < \infty$. We have

$$S_{N(t)+1} > t$$

$$\Rightarrow \mu(m(t)+1) > t$$

$$\Rightarrow \liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$$

Define Truncated rv's $X_n^{(M)}$ by

$$\begin{aligned} X_n^{(M)} &= X_n, \; X_n \leq M \\ &= M, \; X_n > M \end{aligned}$$

and form the new renewal process $N^{(M)}(t)$ using them. Clearly

$$S_{N^{(M)}(t)+1} \leq t + M$$

so that

$$\mu_M(m^{(M)}(t)+1) \le t + M \,,$$

where $\mu_M = E(X_n^{(M)})$ and $m^{(M)}(t) = E(N^{(M)}(t))$.

$$\therefore \quad \overline{\lim_{t\to\infty}} \frac{m^{(M)}(t)}{t} \le \frac{1}{\mu_M}$$

Since $N^{(M)}(t) \ge N(t)$ we also have $m^{(M)}(t) \ge m(t)$ and so

$$\overline{\lim} \frac{m(t)}{t} \le \frac{1}{\mu_M}$$

Now let $M \to \infty$ & the MCT to get $\mu_M \uparrow \mu$ and

$$\overline{\lim} \frac{m(t)}{t} \le \frac{1}{\mu}$$

It follows that $\overline{\lim} \frac{m(t)}{t} = \underline{\lim} \frac{m(t)}{t} = 1/\mu$ so that $\frac{m(t)}{t} \to \frac{1}{\mu}$.

The case $\mu = \infty$ is similar.

qed