# Markov Chains

Consider $\{ \underset{\substack{\uparrow \\ \text{rv}}}{X_t} , t \in \underset{\substack{\uparrow \\ \text{index set}}}{T} \}$ where $T \subset \mathbb{R}$.

In fact $T$ will be a set of time points.
There are two cases of interest: $T$ is "discrete" & $T$ is "cts". For example
$\underbrace{T = \{ t \mid t \geq 0 \}}_{\text{a cts case}}$ or $\underbrace{T = \{ 0, 1, 2, \cdots \}}_{\text{a discrete case}}$

Consider $\bigcup_t \{ \text{possible values of } X_t \}$. This is called the <u>state space</u>. The Markov property states that $\forall$ $t_1 < t_2 < \cdots$

$$\underset{\substack{\uparrow \\ \text{future}}}{X_{t_{n+1}}} \mid \underset{\substack{\uparrow \\ \text{present}}}{X_{t_n}}, \cdots, X_{t_1} \overset{d}{=} X_{t_{n+1}} \mid X_{t_n}$$

$$\Longleftrightarrow E[g(X_{t_{n+1}}, X_{t_{n+2}}, \cdots) \mid X_{t_n}, \cdots, X_{t_1}] = E[g(X_{t_{n+1}}, \cdots) \mid X_{t_n}], \text{ "}\forall\text{" } g$$

## The Chapman-Kolmogorov Eq'n (CKE)

**Theorem** If $\{X_t \mid t \in T\}$ is Markov then $\forall\, t_1 < t_2 < t_3$

$$E[h(X_{t_3}) \mid X_{t_1}] = E\{E[h(X_{t_3}) \mid X_{t_2}] \mid X_{t_1}\}, \text{ "}\forall h\text{"}$$

**Proof.** We have

$$E[h(X_{t_3}) \mid X_{t_1}]$$
$$= E\{E[h(X_{t_3}) \mid X_{t_2}, X_{t_1}] \mid X_{t_1}\}$$
$$= E\{E[h(X_{t_3}) \mid X_{t_2}] \mid X_{t_1}\}, \text{ by}$$

the Markov property

qed

**Transition probabilities for Markov Chains**

$$p_{ij}(t_1, t_2) = P(X_{t_2} = j \mid X_{t_1} = i)$$

<u>Note</u> For Markov Chains the state space is countable & we can label the states $1, 2, \cdots$

Set
$$p_{ij}(u) = p_{ij}(t, t+u)$$
If this does not depend on $t$ then we are in the time homogeneous case. This conditional probability is called a "$u$-step" transition probability. We can also form the matrix of these probabilities

$$\{p_{ij}(u)\}_{i,j}$$

transition matrix

Set $P = \{p_{ij}(1)\}_{i,j}$ & $P(k) = \{p_{ij}(k)\}_{i,j}$

& $R(t) = \begin{pmatrix} P(X_t = 1) \\ P(X_t = 2) \\ \vdots \end{pmatrix}$ $\{= \text{pf of } X_t\}$.

$R(0)$ is the pf of the initial dist'n.

Clearly the CKE in the discrete time case ($T = \{0, 1, 2, \ldots\}$) are

$$P(n+m) = P(n)\, P(m)$$

$$\Rightarrow \quad P(n) = P^n$$

Also

$$\underset{\sim}{p}(n)' = \underset{\sim}{p}(n-1)'\, P$$

$$\Rightarrow \underset{\sim}{p}(n)' = \underset{\sim}{p}(0)'\, P^n$$

$\lim_{n \to \infty} \underset{\sim}{p}(n)$ ?

If this exists, call it $\underset{\sim}{\Pi}$ say, then

$$\underset{\sim}{\Pi}' = \underset{\sim}{\Pi}'\, P$$

so that $\underset{\sim}{\Pi}$ is an eigenvector.

Recall

- state $j$ is accessible from state $i$ if $p_{ij}(n) > 0$ for some $n \geq 0$.

  <u>Notation</u> $i \to j$

- $i$ & $j$ communicate if $i \to j$ & $j \to i$.

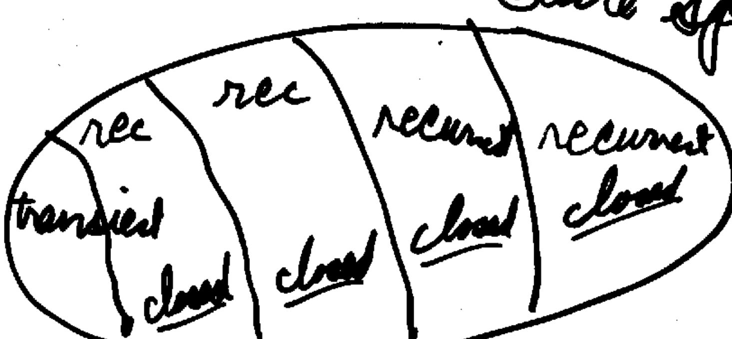
  <u>Notation</u> $i \leftrightarrow j$

<u>Proposition</u> $i \leftrightarrow i$, $i \leftrightarrow j \Rightarrow j \leftrightarrow i$, $i \leftrightarrow k$ & $k \leftrightarrow j$ then $i \leftrightarrow j$.

<u>Proof</u> Obvious

$\leftrightarrow$ partitions the state space

State space




Within each member of the partition all states communicate.

All states communicate $\Leftrightarrow$ MC is <u>irreducible</u>

$i$ is recurrent if starting from $i$ you are certain to return. Let $T_i$ = time it takes to return. If $E(T_i) < \infty$ then we call the state positive recurrent. A recurrent state with $E(T_i) = \infty$ is null recurrent.

If not recurrent then transient.

The period of a state $i$ is $\gcd\{n \mid p_{ii}(n) > 0\}$. Denote this by $d(i)$. If $d(i) = 1$ then we call $i$ aperiodic. Otherwise it is periodic.

Let $T_{ij}$ = 1st time one enters state $j$ starting from $i$ (1st passage time). $T_{ii}$ is our $T_i$. Set $f_{ij}(n) = P(T_{ij} = n)$ to be the "pf" of $T_{ij}$. Note $\sum_n f_{ij}(n)$ may be $< 1$ so we may have an improper distn

Remarks: $T_{ij}$ may be $+\infty$ with $>0$ prob. For recurrent states $T_{ii} \overset{wp1}{<} \infty$. For transient states $P(T_{ii} = \infty) > 0$.

2 Call $f_{ij}(0)$ 0, $\forall i, j$.

Look at

$$\underbrace{P(T_{ij} < \infty)}_{f_{ij}} = \sum_{n=1}^{\infty} f_{ij}(n)$$

Call the generating functions of $\{p_{ij}(n) : n = 0, 1, \ldots\}$ & $\{f_{ij}(n) : n = 0, 1, \ldots\}$ respectively. So $P_{ij}(s)$ & $F_{ij}(s)$

$$P_{ij}(s) = \sum_{n} p_{ij}(n) s^n, \quad F_{ij}(s) = \sum_{n} f_{ij}(n) s^n$$

Note When $T_{ij} \overset{wp1}{<} \infty$ then $F_{ij}(s) = E(s^{T_{ij}})$ = pgf of $T_{ij}$.

**Theorem** $P_{ij}(s) = \delta_{ij} + F_{ij}(s)\, P_{jj}(s)$

($\delta_{ij} = 1$ if $i = j$ & $0$ if $i \neq j$)

**Proof** Suppose $X_0 = i$. Then

$$\sum_{m=0}^{\infty} I_{\{X_m = j\}}\, s^m = \delta_{ij} + s^{T_{ij}} \sum_{m=0}^{\infty} I_{\{X_{m+T_{ij}} = j\}}\, s^m$$

ind

Now take E's to get

$$P_{ij}(s) = \delta_{ij} + F_{ij}(s)\, P_{jj}(s)$$

**Notation** $P_{jj}(1) = \lim_{s \uparrow 1} P_{jj}(s)$

**Theorem** State $j$ is recurrent $\Leftrightarrow P_{jj}(1) = \infty$

**Proof** $j$ is recurrent $\Leftrightarrow F_{jj}(1) = 1$

Since
$$F_{jj}(s) = \frac{P_{jj}(s) - 1}{P_{jj}(s)}$$

we get $P_{jj}(s) \to \infty$ as $s \uparrow 1$. That is $P_{jj}(1) = \infty$.

qed