Theorem $P_{ij}(s) = \delta_{ij} + F_{ij}(s)\, P_{jj}(s)$

($\delta_{ij} = 1$ if $i = j$ & 0 if $i \neq j$)

Proof Suppose $X_0 = i$. Then

$$\sum_{m=0}^{\infty} I_{\{X_m = j\}}\, s^m = \delta_{ij} + s^{T_{ij}} \sum_{m=0}^{\infty} I_{\{X_{m+T_{ij}} = j\}}\, s^m$$

ind

Now take E's to get

$$P_{ij}(s) = \delta_{ij} + F_{ij}(s)\, P_{jj}(s)$$

Notation $P_{jj}(1) = \lim_{s \uparrow 1} P_{jj}(s)$

Theorem State j is recurrent $\Leftrightarrow P_{jj}(1) = \infty$

Proof j is recurrent $\Leftrightarrow F_{jj}(1) = 1$

Since
$$F_{jj}(s) = \frac{P_{jj}(s) - 1}{P_{jj}(s)}$$

we get $P_{jj}(s) \to \infty$ as $s \uparrow 1$. That is $P_{jj}(1) = \infty$.

qed

Suppose j is recurrent. If $i \neq j$ then

$$P_{ij}(s) = F_{ij}(s)\, P_{jj}(s)$$

$\Rightarrow$ $P_{ij}(1) = \infty$, that is

$$\sum p_{ij}^{(n)} = \infty$$

& this is true $\forall i$ such that $\underbrace{f_{ij}}_{F_{ij}(1)} > 0$.

This says that j recurrent & $i \to j$

$\Rightarrow \sum p_{ij}^{(n)} = \infty$

Set $\pi_j = \lim_{s \uparrow 1} (1-s)\, P_{jj}(s)$

We have

$$\frac{1}{(1-s) P_{jj}(s)} = \frac{1 - F_{jj}(s)}{1-s}$$

If j recurrent then $F_{jj}(1) = 1$ & so

$$\frac{1 - F_{jj}(s)}{1-s} \longrightarrow F_{jj}'(1) = E(T_j)$$

Notice $E(T_j) < \infty$ (ie j is positive recurrent)

$$\Leftrightarrow \pi_j = \lim_{s \uparrow 1} (1-s)\, P_{js}(s) > 0$$

<u>Remarks</u> (1) $P_{ij}(s) = \sum_n p_{ij}(n) s^n \Rightarrow P(s) = \sum_n P^n s^n$

which is just $(I - sP)^{-1}$

(2) We will show that a recurrent state i is null $\Leftrightarrow p_{ii}(n) \to 0$ (i transient $\Rightarrow p_{ii}(n) \to 0$)

<u>Proposition</u> If the process is irreducible then for any j, k, h, i $\exists\, c > 0$ st

$$P_{jk}(s) \geq c\, P_{hi}(s), \quad \forall s \in [\rho, 1)$$

where $\rho \in (0,1)$.

<u>Proof</u>: Let $\rho \in (0,1)$ & $s \in [\rho, 1)$. Since

$$P(s) = (I - sP)^{-1}$$

we have

$$(I - sP)\, P(s) = P(s)\, (I - sP) = I$$

That is

$$P(s) - s P P(s) = P(s) - s P(s) P = I$$

$\therefore \quad P(s) = I + sPP(s) = I + sP(s)P$

$\Rightarrow \quad P(s) \geq \rho PP(s) = \rho P(s) P$ {elementwise}

$\Rightarrow \quad P(s) \geq \rho^{m+n} P^m P(s) P^n$

In particular,

$$P_{jk}(s) \geq \rho^{m+n} p_{jh}(m) P_{hi}(s) p_{ik}(n)$$

$$= \left(\rho^{m+n} p_{jh}(m) p_{ik}(n)\right) P_{hi}(s)$$

Now choose m & n so that $p_{jh}(m)\, p_{ik}(n) > 0$ to obtain the result (this can be done since all states communicate).

qed

Theorem If $i \leftrightarrow j$ then

(a) i & j have the same period

(b) i transient $\Rightarrow$ j transient

(c) i positive recurrent $\Rightarrow$ j positive recurrent

Proof: For (b)

Suppose $i \leftrightarrow j$ & i is transient. Then $\exists\ m, n \geq 0$ st

$$\alpha = p_{ij}(m)\, p_{ji}(n) > 0$$

But

$$p_{ij}(m)\, p_{jj}(k)\, p_{ji}(n) \leq p_{ii}(m+k+n)$$

$$\Rightarrow \sum_k \alpha\, p_{jj}(k) \leq \sum_k p_{ii}(m+k+n) < \infty$$

$$\Rightarrow \sum_k p_{jj}(k) < \infty$$

so that j is transient

For (c)

Now suppose i is positive recurrent.

Since $i \leftrightarrow j$ we can find $\alpha > 0$ st for some $\rho < 1$

$$P_{jj}(s) \geq \alpha\, P_{ii}(s) \quad , s \in [\rho, 1)$$

Since i is positive recurrent

$$\lim_{s\uparrow 1}(1-s)\,P_{ii}(s)>0$$

so that

$$\lim_{s\uparrow 1}(1-s)\,P_{jj}(s)\geq \lim_{s\uparrow 1}(1-s)\,\alpha\, P_{ii}(s)>0$$

and hence j is positive recurrent.

<u>For (a)</u>

Use the CKE's to write for suitable m, n

$$P_{ii}^{(m+k+n)}\geq \underbrace{P_{ij}^{(m)}}_{>0}\,P_{jj}^{(k)}\,\underbrace{P_{ji}^{(n)}}_{>0}$$

Hence $p_{ii}^{(m+k+n)} \geq \alpha\, p_{jj}^{(k)}$,

where $\alpha = p_{ij}^{(m)} p_{ji}^{(n)} > 0$. Let

i have period $d(i)$ and j

period $d(j)$. We have

$$p_{ii}^{(m+n)} \geq \alpha\, p_{jj}^{(0)} > 0$$

so that $m+n$ is a multiple

of $d(i)$ - use the notation $d(i) \mid (m+n)$.

If $d(i) \nmid k$ then $d(i) \nmid m+n+k$

so that $p_{ii}^{(m+k+n)} = 0$ &

hence $p_{jj}^{(k)} = 0$. It follows that

$d(i)$ divides any k st $p_{jj}^{(k)} > 0$ &

hence $d(i) \mid d(j)$. Just reverse i & j

to get $d(j) \mid d(i)$ & so $d(i) = d(j)$ qed

Proposition If the process is irreducible and recurrent then for any j, k

$$F_{jk}(1) = 1$$

and

$$\lim_{s \uparrow 1} \frac{P_{jk}(s)}{P_{kk}(s)} = 1$$

Remark $F_{jk}(1) = 1$ is the condition for certain passage from j to k.

Proof We have

$$P_{jk}(s) = \delta_{jk} + F_{jk}(s)\, P_{kk}(s)$$

or in matrix form

$$P(s) = I + F(s)\, D(s) \quad ,$$

where $D(s)$ is the diagonal matrix with kth diagonal element $P_{kk}(s)$. Since $P(s) = (I - sP)^{-1}$ this reduces to

$$(I - sP)^{-1} = I + F(s)\, D(s)$$

$$\Leftrightarrow \quad I = (I - sP) + F(s)\, D(s) - sP\, F(s)\, D(s)$$

$$\Leftrightarrow \quad sP = F(s)(I - sP)\, D(s)$$

or

$$F(s)(I - sP) = sP\, D(s)^{-1}$$

Now since the process is recurrent

$$D(s)^{-1} \to 0 \quad \text{as } s \uparrow 1$$

so that

$$F(1) = F(1)\, P \ (= P\, F(1))$$

It follows that

$$F(1) = P^n\, F(1) \quad , \forall n \geq 0$$

In particular

$$f_{kk} = \sum_j p_{kj}^{(n)} f_{jk} ,$$

where $f_{jk} = F_{jk}(1)$ is the probability of ultimate passage to k starting from j.

Since $f_{kk} = 1$, $0 \leq f_{jk} \leq 1$ & $\sum_j p_{kj}^{(n)} = 1$

we conclude $f_{jk} = 1$ for any $p_{kj}^{(n)} > 0$.

Since $p_{kj}^{(n)} > 0$ for some n we get $f_{jk} = 1$ $\forall j$.

If $j \neq k$ we have

$$F_{jk}(s) = \frac{P_{jk}(s)}{P_{kk}(s)}$$

& since $F_{jk}(1) = f_{jk} = 1$ we conclude

$$\frac{P_{jk}(s)}{P_{kk}(s)} \to 1 \quad \text{as } s \uparrow 1$$

qed

Def'n If $\underset{\sim}{\pi} \geq \underset{\sim}{0}$, $\underset{\sim}{\pi}'\underset{\sim}{1} = 1$ & $\underset{\sim}{\pi}' = \underset{\sim}{\pi}' P$ then $\underset{\sim}{\pi}$ is called a stationary distribution of the chain.

The Chapman-Kolmogorov equations

$$P(n+m) = P(n)\,P(m)$$

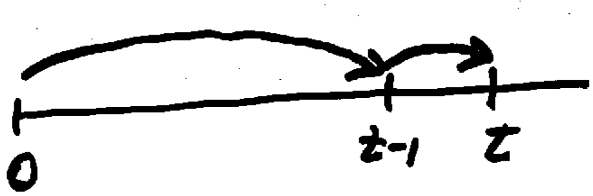
have as special cases

$$P(t) = P\,P(t-1) \qquad \text{— backward}$$

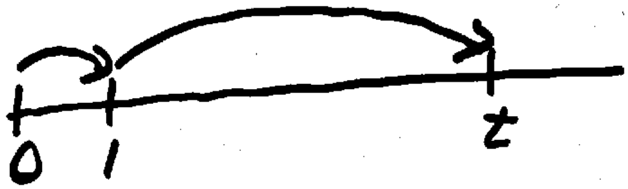
$$P(t) = P(t-1)\,P \qquad \text{— forward}$$

which are the Kolmogorov backward and forward equations respectively.

forward



backward



The forward equations are

$$p_{jk}(t) = \sum_i p_{ji}(t-1)\,p_{ik}$$

If $P_{jk}(t) \to \pi_k$ (not depending on j)
then the forward equations tell us
that $\underline{\pi}$ is a stationary distribution
(invariant measure). In this case
the process is termed ergodic (this phrase
is used in more general contexts).

eg Imagine a particle being in one of
two situations (locations, energy levels,
etc...) & set

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

α & β are the probabilities of
transition between the two states.

"Clearly"

$$P^t = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} + \frac{(1-\alpha-\beta)^t}{\alpha+\beta}\begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix}$$

If $|1-\alpha-\beta|<1$ then

$$P^t \longrightarrow \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix}$$

so that $\pi_1 = \frac{\beta}{\alpha+\beta}$ & $\pi_2 = \frac{\alpha}{\alpha+\beta}$.

If $\alpha = \beta = 0$ then $1-\alpha-\beta = 1$ &

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Here the limiting distribution does depend on the initial state $\underset{\sim}{\Pi}_0$. In fact

$$\underset{\sim}{p}(t) = \underset{\sim}{\Pi}_0 \; (= \underset{\sim}{p}(0))$$

In addition there are ∞ many stationary dist'ns $\underset{\sim}{\Pi}$. The case $\alpha = \beta = 1$ is

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and there is no limiting dist'n but $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ is a stationary dist'n (the only one). This is the periodic case. The final case $(\alpha = \beta = 0)$ is the reducible case.

~

<u>Theorem</u> In the irreducible, positive recurrent case

(1) $\pi_k = \lim_{s \uparrow 1} (1-s)\, P_{jk}(s) > 0$

(2) $\underset{\sim}{\pi}$ is the unique stationary dist'n.

<u>Proof:</u> (1) follows from $\dfrac{P_{jk}(s)}{P_{kk}(s)} \to 1 \quad \uparrow$

our previous results.

For (2) we note

$$P(s) = (I - sP)^{-1}$$

$$\Rightarrow (1-s)\left[\, P(s) - s\, P(s) P \,\right] = (1-s)\, I$$

$$\Rightarrow (1-s)\, P_{kk}(s) - s \sum_j (1-s)\, P_{kj}(s)\, p_{jk} = 1-s$$

$$\overset{s\uparrow 1}{\Longrightarrow} \pi_k - \sum_j \pi_j\, p_{jk} = 0$$

$$\Rightarrow \underset{\sim}{\pi}' = \underset{\sim}{\pi}'\, P$$

Also

$$P(s) = \sum_n P^n s^n$$

Since $P^n \underset{\sim}{1} = \underset{\sim}{1}$ & $\sum s^n = 1/1-s$ we obtain

$$(1-s)\, P(s)\, \underset{\sim}{1} = \underset{\sim}{1}$$

Now $(1-s)\, P(s) \to (\pi_1 \underset{\sim}{1} \;\; \pi_2 \underset{\sim}{1} \; \cdots)$

so that

$$(1-s)\, P(s)\, \underset{\sim}{1} \to (\pi_1 + \pi_2 + \cdots)\, \underset{\sim}{1}$$

and hence $\pi_1 + \pi_2 + \cdots = 1$. We conclude that $\underset{\sim}{\pi}$ is a stationary dist'n. If $\underset{\sim}{a}$ is any other stationary dist'n we have

$$\underset{\sim}{a}' = \underset{\sim}{a}' P = \underset{\sim}{a}' P^n$$

$$\Rightarrow \sum_n \underset{\sim}{a}' s^n = \sum_n \underset{\sim}{a}' P^n s^n$$

$$\Rightarrow \underset{\sim}{a}' = \underset{\sim}{a}'(1-s)\, P(s)$$

$$\overset{s\uparrow 1}{\Longrightarrow} \underset{\sim}{a}' = \underset{\sim}{\pi}' \quad \left\{ \begin{array}{l} \text{as } (1-s)\, P(s) \to (\pi_1 \underset{\sim}{1} \;\; \pi_2 \underset{\sim}{1} \cdots) \\ \text{\& } \underset{\sim}{a}' \underset{\sim}{1} = 1 \end{array} \right.$$

qed

Recall $P_{jj}(s) = 1 + F_{jj}(s)\, P_{jj}(s)$ so that

$$1 - F_{jj}(s) = \frac{1}{P_{jj}(s)}$$

Now
$$P_{jj}(s) = \sum_m p_{jj}(m)\, s^m$$

so that $P_{jj}(1) = \sum_m p_{jj}(m)$. Notice $p_{jj}(0) = 1$ so that $P_{jj}(1) \geq 1$. If j is transient we have $P_{jj}(1) < \infty$ and $F_{jj}(1) = 1 - \frac{1}{P_{jj}(1)} < 1$

which means uncertain ultimate return to j (we already know this). Moreover in this case (j transient)

$$\frac{1}{(1-s)\, P_{jj}(s)} \to \infty \quad , \text{as } s \uparrow 1,$$

since $(1-s)\, P_{jj}(s) \to 0$. If j is recurrent then $F_{jj}(1) = 1$ and

$$\frac{1 - F_{jj}(s)}{1-s} \to F_{jj}'(1)$$

eg For the simple (± 1 jumps with probabilities p & q) random walk on the integers we found that all states were recurrent $\Leftrightarrow p = q = \frac{1}{2}$ (ie the symmetric case). In fact since

$$\frac{1}{\sqrt{1-4x}} = \sum_{k=0}^{\infty} \binom{2k}{k} x^k$$

so that

$$P_{00}(s) = (1-4pq\,s^2)^{-\frac{1}{2}}$$

$$\therefore \quad F_{00}(s) = 1-(1-4pq\,s^2)^{\frac{1}{2}}$$

&

$$F_{00}'(s) = \frac{4pq\,s}{(1-4pq\,s^2)^{3/2}}$$

So $F_{00}(1) = 1-(1-4pq)^{\frac{1}{2}}$

$$= 1-(1-4p(1-p))^{\frac{1}{2}} = 1-((1-2p)^2)^{\frac{1}{2}}$$

$$= 1-|p-q| \quad = 1 \Leftrightarrow p = q$$

Moreover when $p = q = \frac{1}{2}$, $F_{00}'(1) = +\infty$ so that $E(T_{00}) = \infty$ & the process is null recurrent.

Let j be recurrent. Then

$$\frac{1-F_{jj}(s)}{1-s} = \frac{1-\sum_{m=0}^{\infty} P(T_{jj}=m)\, s^m}{1-s}$$

where T_{jj} is the return time ($P(T_{jj}=0)=0$ by convention & the probability is a conditional one – on $X_0=j$). So

$$\frac{1-F_{jj}(s)}{1-s} = \sum_{m=0}^{\infty} \frac{P(T_{jj}=m) - P(T_{jj}=m)\, s^m}{1-s}$$

$$= \sum_{m=0}^{\infty} P(T_{jj}=m)\, \frac{1-s^m}{1-s}$$

$$= \sum_{m=0}^{\infty} P(T_{jj}=m)\, [1+s+\cdots+s^{m-1}]$$

which is increasing in s. It follows that

$$\lim_{s\uparrow 1} \frac{1-F_{jj}(s)}{1-s}$$

exists or is $+\infty$. Hence

$$\pi_j = \lim_{s \uparrow 1} (1-s)\, P_{jj}(s)$$

exists (it may be 0). In fact for any state j π_j is well-defined (in the transient case $\pi_j = 0$). We showed

$$\pi_j > 0 \iff j \text{ is positive recurrent}$$

In addition, if the process is irreducible and positive recurrent then $\underline{\pi}$ is a stationary dist'n. The reverse is (almost) true.

<u>Theorem</u> Suppose the process is irreducible.

(a) If the state space is finite then the process is positive recurrent.

(b) If the process has a stationary distribution then it is positive recurrent

<u>Proof</u>: Exercise

<u>Theorem</u> Let $\{X_n\}$ be irreducible, aperiodic, positive recurrent MC. Then

$$p_{ij}^{(n)} \to \frac{1}{\mu_j}, \text{ as } n \to \infty$$

where μ_j is the mean recurrence time of state j.

<u>Proof</u>: Consider a new MC $\{Y_n\}$ which is ind of $\{X_n\}$ but has the same transition matrix $P = \{p_{ij}\}$.

Now look at the "coupled" process $\{(X_n, Y_n)\}$. This is also a MC with transition probabilities

$$p_{ij,kl}$$
$$= P[(X_n, Y_n) = (k,l) \mid (X_{n-1}, Y_{n-1}) = (i,j)]$$
$$= P(X_n = k \mid X_{n-1} = i)\, P(Y_n = l \mid Y_{n-1} = j)$$
$$= p_{ik}\, p_{jl}$$

Clearly $\{(X_n, Y_n)\}$ is irreducible & aperiodic. It also has a stationary dist'n with elements $\pi_i \pi_j$, where π is the stationary dist'n of $\{X_n\}$ ($\leftarrow$ we know this exists). $\therefore$ $\{(X_n, Y_n)\}$ is positive recurrent. Suppose $(X_0, Y_0) = (i,j)$ & let T be the first passage time to (s,s).

Now for $n \geq T$ we have that the dist'n of X_n & Y_n will be equal!

So conditional on $(X_0, Y_0) = (i, j)$

$$p_{ik}(n) = P(X_n = k)$$

$$= P(X_n = k, T \leq n) + P(X_n = k, T > n)$$

$$= P(Y_n = k, T \leq n) + P(X_n = k, T > n)$$

$$\leq P(Y_n = k) + P(T > n)$$

$$= p_{jk}(n) + P(T > n)$$

Also $p_{jk} \leq p_{ik}(n) + P(T > n)$

$$\Rightarrow |p_{ik}(n) - p_{jk}(n)| \leq P(T > n) \longrightarrow 0$$

Now take the stationary dist'n of $\{X_n\}$, π. We then have

$$\underset{\sim}{\pi}' = \underset{\sim}{\pi}' P = \underset{\sim}{\pi}' P^m$$

$$\Rightarrow \pi_k = \sum_i \pi_i \, p_{ik}(m)$$

$$\Rightarrow \pi_k - p_{jk}(m) = \sum_i \pi_i \left(p_{ik}(m) - p_{jk}(m) \right)$$

$$\Rightarrow |\pi_k - p_{jk}(m)| \le \sum_i \pi_i \underbrace{|p_{ik}(m) - p_{jk}(m)|}_{\le 2}$$

$$\to 0 \qquad \text{(use the DCT)}$$

Hence

$$\lim_{m \to \infty} p_{jk}(m) = \pi_k = \frac{1}{\mu_k} \leftarrow E(T_k)$$

qed

<u>Note</u> This result also shows $P(X_m = k) \to \pi_k$. To see this write

$$P(X_m = k) = \sum_j P(X_m = k \mid X_0 = j) \, P(X_0 = j)$$

$$\xrightarrow{} \sum_j \pi_k \, P(X_0 = j) = \pi_k$$

why?