More on Markov Chains

For discrete time ($T = Z^+$) a (the) stationary dist'n $\underset{\sim}{\pi}$ is any pf satisfying $\underset{\sim}{\pi}' = \underset{\sim}{\pi}' P$.

<u>Note</u> – $\underset{\sim}{\pi}$ is not unique

We know

– j is recurrent iff $P_{jj}(1) = \infty$ (ie $\sum_n p_{jj}(n) = \infty$)

– j recurrent then it is positive recurrent iff

$$\pi_j = \lim_{s \uparrow 1} (1-s) P_{jj}(s) > 0$$

& then $\pi_j = \dfrac{1}{E(T_{jj})}$

Notice that

$$\frac{1 - F_{jj}(s)}{1-s} = \frac{1 - E(s^{T_{jj}})}{1-s} = \frac{1 - \sum_k s^k P(T_{jj} = k)}{1-s}$$

$$= \left[\sum_k P(T_{jj} = k) - \sum_k s^k P(T_{jj} = k)\right] / (1-s)$$

$$= \sum_k P(T_{jj} = k) \frac{1 - s^k}{1-s}$$

$$= \sum_k P(T_{jj} = k) (1 + s + \cdots + s^{k-1})$$

& this increases with s & has lim as $s \uparrow 1$ of $\overbrace{\sum_k P(T_{jj} = k) k}^{E(T_{jj})}$

– process irreducible + positive recurrent

$\Rightarrow \underset{\sim}{\Pi}$ is unique + $> \underset{\sim}{0}$

$$\Pi_j = \frac{1}{E(T_{jj})}$$

If the process is irreducible with a countably ∞ state space then

$\underset{\sim}{\Pi}$ exists $\Rightarrow$ positive recurrent

If the process is irreducible + finite then all states are positive recurrent

$$\left.\begin{array}{l} \text{Irreducible + positive recurrent + aperiodic} \\ \Rightarrow \lim_{n\to\infty} p_{ij}(n) = \Pi_j > 0 \\ \text{+ } \underset{\sim}{\Pi} \text{ is the unique stationary dist'n} \end{array}\right\} \text{proved}$$

For irreducible + aperiodic + null recurrent we do not have a stationary distribution. (By applying renewal theory we in fact know $p_{ij}(n) \to 0$.) From the coupling argument we still have

$$p_{ik}(n) - p_{jk}(n) \to 0 \qquad \forall\, i, j, k \quad (*)$$

We now use this to show $p_{ij}(n) \to 0$.

Now suppose $p_{ij}(n)$ does not $\to 0$ as $n \to \infty$. For fixed i & j $\exists$ a subsequence of the n's so that $p_{ij}(n_k^{(i)}) \to \geq 0$ but not all 0 # say α_j. α_j will not depend on i $\because$ of $(*)$ but the subsequence might. Now let $\epsilon_i \downarrow 0$. For each i choose n_k so that $n_k \in \{n_k^{(i)}\}$ & $p_{ij}(n_k)$ is within ϵ_i of α_j. In addition choose the n_k's to increase with i. We then have

$$p_{ij}(n_k) \to \alpha_j \qquad \text{for each } j$$

& not all α_j are 0. (We have just employed a diagonalization argument.) Let F be a finite set of j states. Then

$$\sum_{j \in F} \alpha_j = \lim_{k \to \infty} \underbrace{\sum_{j \in F} p_{ij}(n_k)}_{\leq \sum_j p_{ij}(n_k) = 1} \leq 1$$

Hence $0 < \sum_j \alpha_j \leq 1$.

Now

$$\underbrace{\sum_{\ell \in F} p_{i\ell}(n_k)\, p_{\ell j}}_{\to \sum_{\ell \in F} \alpha_\ell p_{\ell j}} \leq p_{ij}(n_k+1) = \underbrace{\sum_\ell p_{i\ell}\, p_{\ell j}(n_k)}_{\to \alpha_j}$$

$\therefore \quad \sum_{l \in F} \alpha_l p_{lj} \le \alpha_j$

& so

$$\sum_l \alpha_l p_{lj} \le \alpha_j \quad , \forall j$$

In fact equality must hold $\forall j$ for if we had $<$ for some j then

$$\sum_l \alpha_l = \sum_l \alpha_l \underbrace{\sum_j p_{lj}}_{1} = \sum_{l,j} \alpha_l p_{lj} = \sum_{j,l} \alpha_l p_{lj} < \sum_j \alpha_j$$

which is impossible. $\therefore$

$$\sum_l \alpha_l p_{lj} = \alpha_j \quad , \forall j$$

Now set $\alpha = \sum_j \alpha_j$ & $\pi_j = \alpha_j / \alpha$. We then have $\underset{\sim}{\pi}$ is a pf & $\underset{\sim}{\pi}' = \underset{\sim}{\pi}' P$. But this cannot be. It follows that

$$p_{ij}(n) \to 0 \quad , \forall i j$$

<u>Irreducible + aperiodic + transient</u> — obvious that $p_{ij}(n) \to 0$

<u>Theorem</u> MC irreducible & aperiodic then

$$p_{ij}(n) \to \frac{1}{E(T_{jj})}$$

Assume irreducible + aperiodic

Let $i, j \in S$. We want to show $p_{ij}(n) > 0$ $\forall$ n large enough.

Sol'n Since j is aperiodic $\exists$ $m_1, \dots, m_r$ with $\gcd = 1$ & $p_{jj}(m_s) > 0$ for $1 \le s \le r$. Now $\exists$ M $\ni$ for $m \geq M$ we can write $m = \sum_{s=1}^{r} a_s m_s$, $a_s \in \mathbb{Z}^+$

Now use the CKE to get

$$p_{jj}(m) \geq \prod_{s=1}^{r} p_{jj}(m_s)^{a_s} > 0 .$$

Hence $p_{jj}(m) > 0$ $\forall$ $m \geq M$.

Now since $i \leftrightarrow j$ choose k so that $p_{ij}(k) > 0$.

Then $\quad p_{ij}(m+k) \geq p_{ij}(k)\, p_{jj}(m) > 0 \quad$ if $m \geq M$

Back to coupling

Let $\{X_n\}$ + $\{Y_n\}$ be iid MC's with Transition matrix P which are irreducible + aperiodic. Set

$$\underset{\sim}{Z}_n = (X_n, Y_n)$$

Let $i, j, k, l \in S$. Let $N(i,j)$ + $N(k,l)$ be st

$$p_{ij}(n) > 0 \quad , \forall n \geq N(i,j)$$
$$p_{kl}(n) > 0 \quad , \forall n \geq N(k,l)$$

Note that both $N(i,j)$ & $N(k,l)$ exist from the previous problem.

We now have $\forall\, n \geq \max\{N(i,j), N(k,l)\}$

$$P\left(\underset{\sim}{Z}_n = (k,l) \mid \underset{\sim}{Z}_0 = (i,j)\right) = p_{ik}(n)\, p_{jl}(n) > 0$$

so that $\{\underset{\sim}{Z}_n\}$ is irreducible & aperiodic.

<u>Remark</u> Take $S = \{1,2\}$ & $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Then $\{(1,1), (2,2)\}$ & $\{(1,2), (2,1)\}$ are closed sets of states for $\{\underset{\sim}{Z}_n\}$.

Obviously $\{\underset{\sim}{Z}_n\}$ is <u>reducible</u>. The reason this happens is that $\{X_n\}$ is periodic (it is irreducible)

Once we have $\{\underset{\sim}{Z}_n\}$ irreducible & aperiodic we follow the coupling proof to get $(*)$ which then leads to $p_{ij}(n) \to 0 = \dfrac{1}{E(T_{jj})}$ $\leftarrow \infty$

Sums of rv's

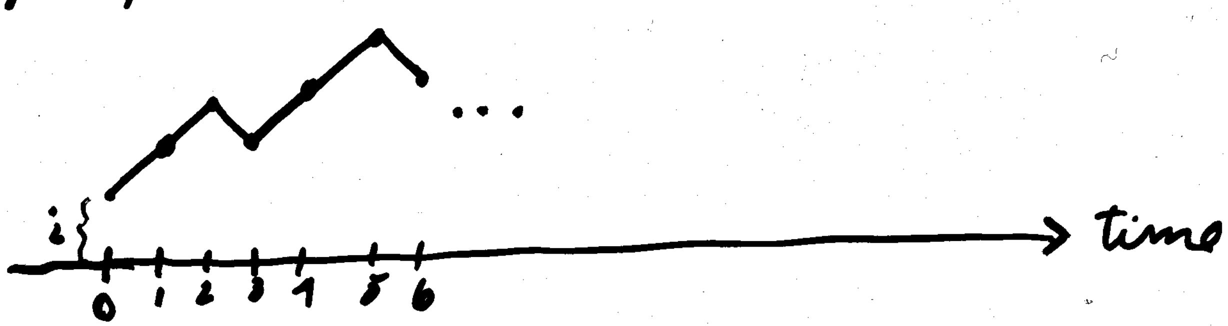
Let $X_0, X_1, \cdots$ be rv's & set

$$S_t = X_0 + X_1 + \cdots + X_t$$

The process $\{S_t \,;\, t = 0, 1, \cdots\}$ is called a random walk. Typically, $X_1, X_2, \cdots$ are ind/iid while $S_0 = X_0$ is treated as a constant & is the starting point of the walk. In the independent case $\{S_t\}$ is a Markov process & if we add the id assumption it is time homogeneous.

The simple random walk on Z

Here, $X_1, X_2, \cdots$ are iid taking values ± 1 with probabilities $p = P(X_1 = 1)$ & $q = P(X_1 = -1)$ with $0 < p < 1$. Set the starting point $S_0 = i$. A typical sample path would look like



We have seen that this Markov Chain is irreducible and is recurrent in the symmetric case ($p=q=\frac{1}{2}$), however the process is null recurrent in the sense that average return times are infinite. When $p \neq q$ then the process is transient. Set $Y = \sum_{t=1}^{\infty} I_{A_t}$, where $A_t = \{S_t = i\}$. In the recurrent case $Y \overset{as}{=} +\infty$. In the transient case Y is geometric with $E_i(Y) < \infty$ (the i is to remind us that we are conditioning on the initial state. We will drop it when this point is clear.). So

$$P(Y=\infty)=1 \Leftrightarrow E(Y) = \sum P(A_t) = \infty \left\{ \begin{matrix} \Leftrightarrow p=q=\frac{1}{2} \\ \Leftrightarrow E(X_1)=0 \end{matrix} \right\}$$

Notice $\{Y=\infty\} = \{A_t \text{ io}\}$ so that we have

$$P(A_t \text{ io}) = 1 \Leftrightarrow \sum P(A_t) = \infty$$

This reminds us of the Borel Cantelli Lemma & 0-1 Law. Certainly $\sum P(A_t) < \infty \Rightarrow P(A_t \text{ io}) = 0$ but note the A's are not typically independent. It is tempting to approach the problem via tail events, but is $\{A_t \text{ io}\}$ one of them?

For random walks with general iid steps X_k the recurrence problem is more complicated (see Breiman's text Probability, section 3.7). We will only consider integer steps X_k. Without loss of generality we take $S_0 = X_0 = 0$ so that we are then interested in the event $\{S_n = 0 \text{ io}\}$. We have

<u>Theorem</u> $E(X_k) = 0 \Rightarrow P(S_n = 0 \text{ io}) = 1$

<u>Proof</u> For any state j let $T \geq 1$ be the first time $S_n = j$ and set

$$N_j = \sum_{n=1}^{\infty} I\{S_n = j\}$$

Then

$$E(N_j) = E\left[E(N_j | T)\right] = \sum_{t=1}^{\infty}\left[\sum_{n \geq t} P(S_n = j | T = t)\right] P(T = t)$$

(since $n < T \Rightarrow S_n \neq j$)

$$= \sum_{t=1}^{\infty} P(T = t)\left[1 + \sum_{n=t+1}^{\infty} P(S_n = j | T = t)\right]$$

$$= \sum_{t=1}^{\infty} P(T=t)\,[1+E(N_0)]$$

$$= P(T<\infty)\,[1+E(N_0)]$$

$$\le 1+E(N_0)$$

Now take $\epsilon>0$. For integer $M>0$ we have

$$\sum_{n=1}^{\infty} P(|S_n|\le M) = \sum_{n=1}^{\infty}\sum_{j=-M}^{M} P(S_n=j)$$

$$= \sum_{j=-M}^{M}\sum_{n=1}^{\infty} P(S_n=j) = \sum_{j=-M}^{M} E(N_j)$$

$$\le (2M+1)\,[1+E(N_0)]$$

By the SLLN

$$\frac{S_n}{n} \xrightarrow{as} 0 \Rightarrow \frac{|S_n|}{n} \xrightarrow{as} 0 \Rightarrow \frac{|S_n|}{n} \xrightarrow{P} 0$$

so that $P(|S_n|\le\epsilon n)\to 1$. Now take n_0 large enough so that $n\ge n_0$ implies $P(|S_n|\le\epsilon n)\ge\frac{1}{2}$ say.

In particular, for M large enough we have

$$P(|S_m| \le M) \geq \tfrac{1}{2} \quad \text{for } m_0 \le m \le \frac{M}{\epsilon}$$

so that

$$1 + E(N_0) \geq \frac{1}{2M+1} \sum_{m=1}^{\infty} P(|S_m| \le M)$$

$$\geq \frac{1}{2M+1} \sum_{m_0 \le m \le \frac{M}{\epsilon}} P(|S_m| \le M)$$

$$\geq \frac{1}{2M+1} \sum_{m_0 \le m \le \frac{M}{\epsilon}} \frac{1}{2}$$

$$\geq \frac{1}{(2M+1)2} \left(\frac{M}{\epsilon} - m_0 \right)$$

$$\to \frac{1}{4\epsilon} \quad \text{as } M \to \infty$$

It then follows (since $\epsilon > 0$ is arbitrary) that $E(N_0) = \infty$ qed

The Ballot Problem ($\simeq$ 1850)

2 candidates in an election A & B. Suppose A wins. Let α = # of votes for A & β = # for B.

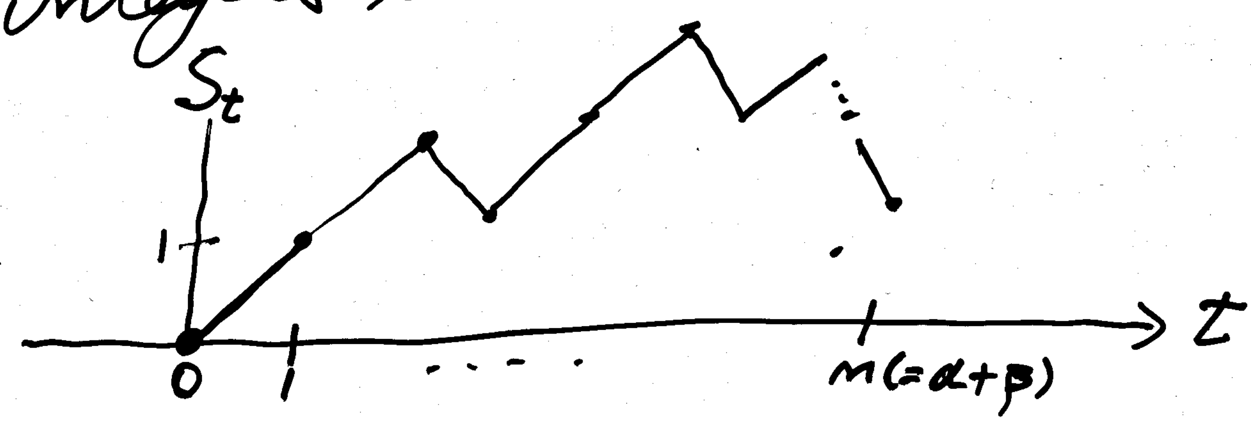
P(A was <u>always</u> ahead of B)

Assign +1 to a vote for A & -1 to a vote for B. As the votes come in you have

$$0, X_1, X_2, \cdots, X_n \qquad (n = \alpha + \beta)$$

Let $S_t = 0 + X_1 + \cdots + X_t$ then

$\{S_t\}$ is a random walk on the integers which starts at 0.



The solution to the problem is to calculate $P(S_t > 0 \text{ for } 0 < t \leq n)$ when starting from 0. In order to solve this consider our simple random walk $\{S_t\}$ starting from a say (ie $S_0 = a$). Suppose we wish to calculate $P(S_n = b)$. Consider

$$a \to s_1 \to s_2 \to \cdots \to s_n = b$$

n arrows

Let r = # of steps to the right (ie # of $+1$'s)
l = # of steps to the left

Then $r + l = n$ & $r - l = b - a$ if we are to end up at b at time n. Hence

$$r = \frac{n+b-a}{2}, \quad l = \frac{n-b+a}{2}$$

so that

$$P(S_n = b) = \binom{n}{\frac{n+b-a}{2}} p^{\frac{n+b-a}{2}} q^{\frac{n-b+a}{2}}$$

# of paths from a to b

req'd → (with $\frac{n+b-a}{2}$ steps to the right)

Suppose now that both $a, b > 0$. Set

$N_m(a,b) = \#$ of paths from a to b

& $N_m^0(a,b) = \#$ of paths from a to b which hit the time axis at least once

Every path from a to b which crosses the t-axis at least once has a reflection about the t-axis which is a path from $-a$ to b. The reverse is also true so that

$$N_m^0(a,b) = N_m(-a,b)$$

We have just employed the reflection principle.

We are now in a position to solve the ballot problem. So, let $b > 0$ & $a = 0$. The # of paths from $a=0$ to b which do not revisit the t-axis is

$$\frac{b}{m} N_m(0,b)$$

To see this note that every such path first goes to 1 (more specifically $(1,1)$). Hence the number of desired paths is

$$N_{m-1}(1,b) - N_{m-1}^0(1,b) = N_{m-1}(1,b) - N_{m-1}(-1,b)$$

Now, $N_m(a,b) = \binom{m}{\frac{m+b-a}{2}}$ so that the req'd # is

$$\binom{m-1}{\frac{m-1+b-1}{2}} - \binom{m-1}{\frac{m-1+b+1}{2}}$$

$$= \binom{m-1}{\frac{m+b}{2}-1} - \binom{m-1}{\frac{m+b}{2}} = \binom{m-1}{k-1} - \binom{m-1}{k}, \quad k = \frac{m+b}{2}$$

$$= \frac{(m-1)!}{(k-1)!\,(m-k)!} - \frac{(m-1)!}{k!\,(m-k-1)}$$

$$= \frac{km(m-1)!}{km(k-1)!\,(m-k)!} - \frac{m(m-1)!\,(m-k)}{m\,k!\,(m-k-1)(m-k)}$$

$$= \frac{k\;m!}{m\;k!\,(m-k)!} - \frac{(m-k)}{m}\,\frac{m!}{k!\,(m-k)!}$$

$$= \binom{m}{k}\frac{k-(m-k)}{m} = \frac{2k-m}{m}\binom{m}{k}$$

$$= \frac{m+b-m}{m}\binom{m}{\frac{m+b}{2}} = \frac{b}{m}\,N_m(0,b)$$

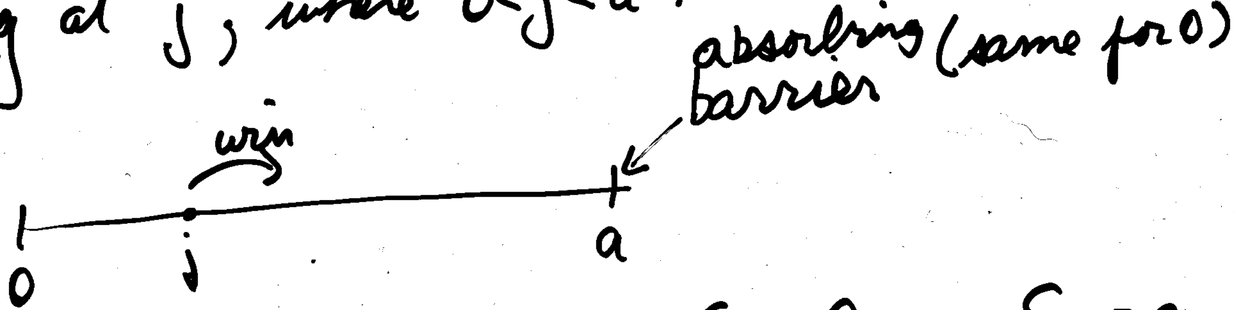
For the ballot problem $m = \alpha+\beta$, $b = \alpha-\beta$ ($\nmid a=0$)
so that

$$P(A \text{ is always ahead}) = \frac{\frac{b}{m}\,N_m(0,b)}{N_m(0,b)} = \frac{b}{m}$$

$$= \frac{\alpha-\beta}{\alpha+\beta}$$

The Gambler's Ruin Problem

We consider a simple random walk $\{S_t\}$ starting at j, where $0<j<a$.



The process stops if either $S_t = 0$ or $S_t = a$.
This Markov Chain has $a+1$ states with
both 0 & a being absorbing states (they are
of course recurrent) and $\{1, \cdots, a-1\}$ being
a transient class. There are 3 classes
to this chain $\{0\}, \{1, \cdots, a-1\}, \{a\}$. The only
nonzero transition probabilities (1-step) are

$$p_{i,i+1} = p \ \& \ p_{i,i-1} = q \quad , \quad 0<i<a$$

Set $\rho_j(t) = P(S_t = 0 \mid S_0 = j)$. This is also denoted
by $p_{j0}(t)$. We have (recall the backward equations)
for $t>0$

$$\rho_j(t) = p\,\rho_{j+1}(t-1) + q\,\rho_{j-1}(t-1) \quad , \quad 0<j<a$$

$$\rho_0(t) = \rho_0(t-1) \ , \ \rho_a(t) = \rho_a(t-1)$$

with boundary conditions

$$\rho_j(0) = \begin{cases} 1 & , j=0 \\ 0 & , j>0 \end{cases}$$

"Clearly" $\rho_j(t)$ increases with t & since it is bounded we have a limit ρ_j called the probability of ultimate ruin. If we let $t \to \infty$ in the above equations we obtain

$$\rho_j = p\,\rho_{j+1} + q\,\rho_{j-1} \quad (0 < j < a)$$

with boundary conditions $\rho_0 = 1$, $\rho_a = 0$. This difference equation is easily solved yielding

$$\rho_j = \frac{(q/p)^j - (q/p)^a}{1 - (q/p)^a}, \quad p \neq q$$

$$= \frac{a-j}{a}, \quad p = q = \frac{1}{2}$$

In the extreme case $a = \infty$ (playing against an infinitely rich opponent!) this reduces to

$$\rho_j = (q/p)^j, \quad p > q$$

$$= 1, \quad p \leq q$$

These results can also be obtained via martingale methods (later).