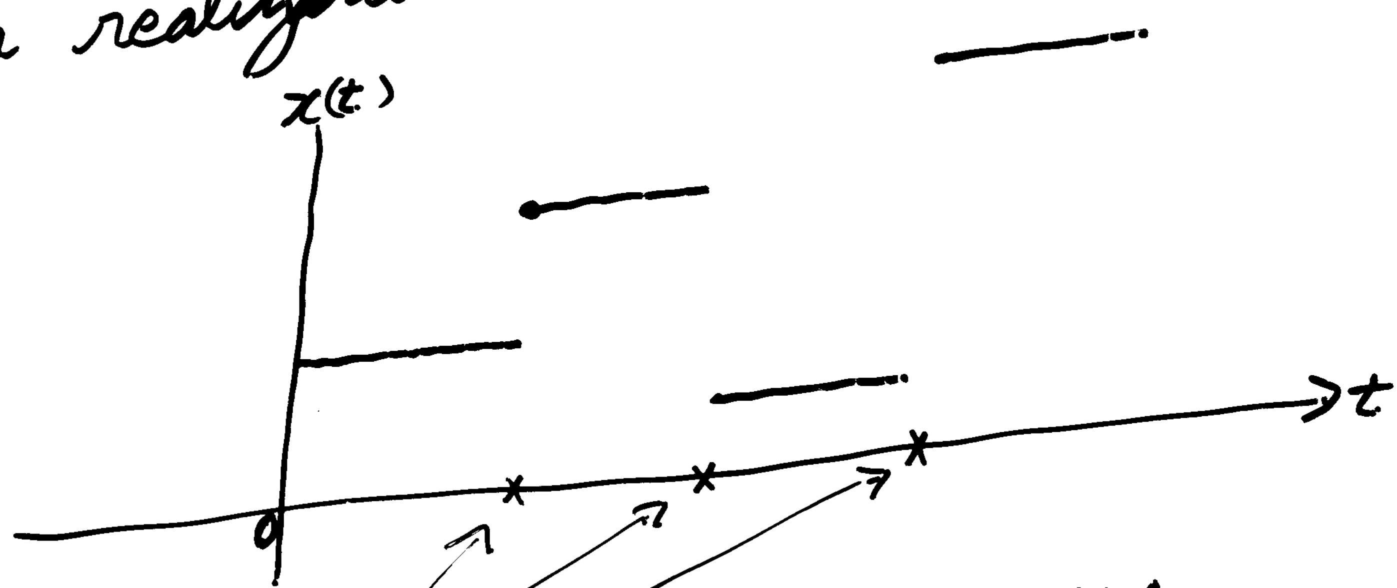
Continuous time processes - Part I

We are going to look at $\{X(t) \mid t \geq 0\}$ such that the state space is $\{0, 1, 2, \dots\}$ & such that $\forall \; t_0 < t_1 < t_2 < \cdots$ $\{X(t_k)\}$ is a MC. We also assume time homogeneous processes. The Markov property is $\Leftrightarrow$ given $X(t)$, $\{X(s) \mid s < t\}$ and $\{X(s) \mid s > t\}$ are independent.

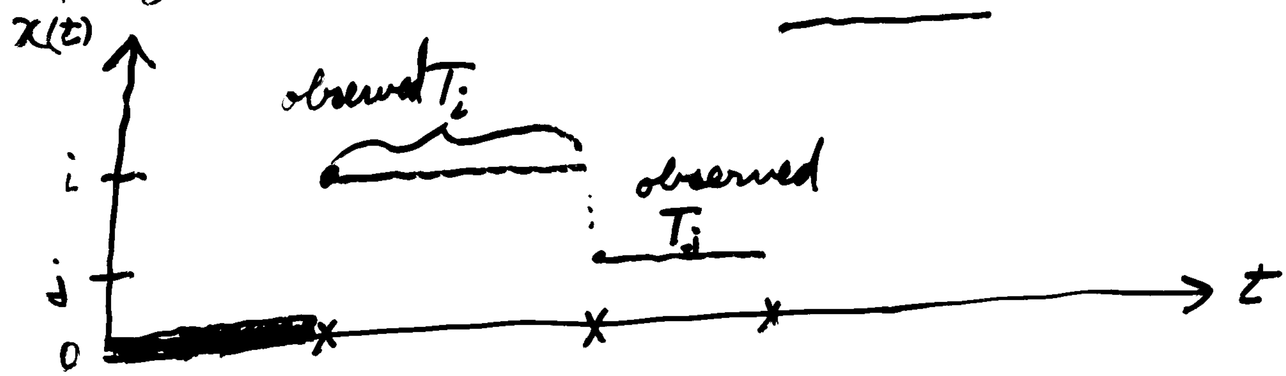
A typical "data set" or sample path or realization looks like



form a point process

A cts time MC $\{X(t) : t \geq 0\}$ may be thought of as a point process where points occur in the following way:

- $X(t)$ stays in state i for a time $T_i \sim$ exponential(q_i)
- the T_i are independent
- when leaving i the probability of going to j is p_{ij} $(j \neq i)$

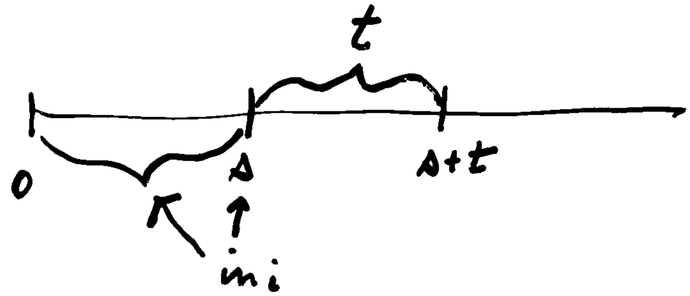


Remarks.

(a) If the process is observed only at jumps then we get a discrete time MC.

(b) A semi-Markov process would be similar except that the times between jumps would not be exponential and the time stayed in i would depend on the target j.

(c) $P(T_i > t+s \mid T_i > s) = P(T_i > t)$ ← lack of memory



$\therefore$ of the Markov Property.

$\because$ $T_i \sim \text{exponential}(q_i)$

Note that large rates ⇒ small means so it's not hard to imagine jumping right out of the state space.

The q_i may be thought of as the rate of leaving i. Typically $0 < q_i < \infty$ but $q_i = 0$ may be thought of as an <u>absorbing state</u> and $q_i = \infty$ an "instantaneous" state — leave as soon as you enter.

The transition rate from i to $j \neq i$ is defined as

$$q_{ij} = q_i P_{ij}$$

The process is <u>regular</u> or <u>honest</u> if there can only be a finite # of transitions in $[0, t]$, $\forall t > 0$. ∞ #? That is in fact a possibility which we will see in a Birth process. Assuming one doesn't escape the state space

$$\sum_{j \neq i} P_{ij} = 1$$

Set $q_{ii} = -q_i$ and $Q = \{q_{ij}\}$.
This matrix is called the generator of the process and plays a similar role to that of the transition matrix P in a discrete time MC. Notice

$$\sum_j q_{ij} = \left(\sum_{j \neq i} q_i p_{ij}\right) - q_i$$

$$= \left(q_i \sum_{j \neq i} p_{ij}\right) - q_i = 0$$

That is

$$Q\underset{\sim}{1} = \underset{\sim}{0}$$

The CKE are

$$P(t+s) = P(t)\,P(s) \quad ,$$

where $P(t) = \{p_{ij}(t)\}$ and $p_{ij}(t)$ is the <u>transition function</u>.

Consider

$$p_{ij}(t+h) = \sum_k P(X(t+h)=j \mid X(h)=k, X(0)=i) \times P(X(h)=k \mid X(0)=i)$$

$$= \sum_k P(X(t+h)=j \mid X(h)=k)\,P(X(h)=k \mid X(0)=i)$$

Note that we have conditioned <u>back</u> to $X(h)$. So

$$p_{ij}(t+h) = \sum_k p_{kj}(t)\,p_{ik}(h)$$

$$= \sum_k p_{ik}(h)\,p_{kj}(t)$$

which is, of course, restating the CKE in component form. In matrix terms

$$P(t+h) = P(h)\,P(t)$$

Now, under certain conditions,

$$Q = \lim_{h \downarrow 0} \frac{P(h) - I}{h} = \dot{P}(0)$$

In this case

$$\frac{P(t+h) - P(t)}{h} = \frac{P(h)\,P(t) - P(t)}{h}$$

$$= \left[\frac{P(h) - I}{h}\right] P(t)$$

so that

$$\dot{P}(t) = Q\,P(t) \qquad \left\{ \text{“}\Rightarrow\text{”} \; P(t) = e^{Qt} \right\}$$

which are the <u>backward equations</u>.

There was an interchange of $\lim_{h \downarrow 0}$ and summation which has to be justified.

For $\{X(t) \mid t \geq 0\}$ the KBE & KFE's are

$$\dot{P}(t) = Q\,P(t)$$

$$\dot{P}(t) = P(t)\,G$$

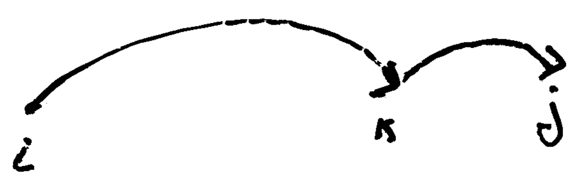
or

$$\dot{p}_{ij}(t) = \sum_{k} q_{ik}\, p_{kj}(t) \qquad \text{— KBE}$$

$$\dot{p}_{ij}(t) = \sum_{k} p_{ik}(t)\, q_{kj} \qquad \text{— KFE}$$

(B)

(F)



$Q = \{q_{ij}\}$ is often denoted by Λ or G.

$i \neq j \quad p_{ij}(h) = q_{ij}\,h + o(h) \qquad q_i \equiv -q_{ii}$

$Q\underset{\sim}{1} = \underset{\sim}{0}$ — <u>often</u>

To see this go back to the component equation

$$p_{ij}(t+h) - p_{ij}(t) = \left(\sum_{k} p_{ik}(h)\, p_{kj}(t)\right) - p_{ij}(t)$$

$$= \left(\sum_{k \neq i} p_{ik}(h)\, p_{kj}(t)\right) + p_{ii}(h)\, p_{ij}(t) - p_{ij}(t)$$

or

$$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \sum_{k \neq i} \frac{p_{ik}(h)}{h}\, p_{kj}(t) + \frac{(p_{ii}(h) - 1)}{h}\, p_{ij}(t)$$

Now

$$\left.\begin{aligned} \frac{p_{ik}(h)}{h} &\to q_{ik} \\ \frac{p_{ii}(h)-1}{h} &\to \underbrace{q_{ii}}_{-q_i} \end{aligned}\right\} \text{ must be proved}$$

Assuming we can interchange the lim & Σ we get

$$\dot{p}_{ij}(t) = \left(\sum_{k \neq i} q_{ik}\, p_{kj}(t)\right) - q_i\, p_{ij}(t)$$

Examples

Poisson process of rate λ on $t \geq 0$

- $N(0) = 0$
- $N(t)$ has independent increments
- $N(t)$ has "local" transitions given by

$$P(N(t+h) = j+1 / N(t) = j) = \lambda h + o(h)$$

$$P(N(t+h) > j+1 / N(t) = j) = o(h)$$

$$P(N(t+h) = j \mid N(t) = j) = 1 - \lambda h + o(h)$$

& this leads to $N(t) \sim \text{Poisson}(\lambda t)$.

We can specify the Poisson model by assuming that the only possible transitions out of state i is to $i+1$ with constant rate λ. Then we have the KE's

$$\dot{p}_{ik}(t) = \lambda\left(p_{i,k-1}^{(t)} - p_{ik}(t)\right) \qquad (F)$$

$$\dot{p}_{ik}(t) = \lambda\left(p_{i+1,k}(t) - p_{ik}(t)\right) \qquad (B)$$

The sol'n to either is

$$p_{ik}(t) = \frac{e^{-\lambda t}(\lambda t)^{k-i}}{(k-i)!} \qquad k \geq i$$

Of course $p_{ik}(t) = 0$ for $k<i$. We can solve these use generating functions

$$P_i(z,t) = \sum_{k=0}^{\infty} p_{ik}(t)\, z^k$$

The (F) + (B) become

$$\dot{P}_i(z,t) = \lambda(z-1)\, P_i(z,t) \quad (F)$$

$$\dot{P}_i(z,t) = \lambda\left(P_{i+1}(z,t) - P_i(z,t)\right) \quad (B)$$

with initial conditions

$$P_i(z,0) = z^i$$

The (F) eq'ns have sol'n

$$P_i(z,t) = z^i e^{\lambda t(z-1)} \quad (*)$$

For our Poisson process $i = 0$ ($\therefore N(0) = 0$) which yields the pgf

$$e^{\lambda t(z-1)}$$

for $N(t)$. This is the pgf of a Poisson(λt).
You can verify that (*) satisfies the (B) eq'ns.

Birth & Death process

$i \to i+1$ with rate λ_i ← birth rate

$i \to i-1$ " " μ_i ← death rate

The simple birth process (Yule) has $\lambda_i = i\lambda$ & if $\mu_i = i\mu$ we get a simple B&D process. If we allow immigration (one at a time) at constant rate r then setting

$$\lambda_i = r + i\lambda, \quad \mu_i = i\mu$$

yields a simple B&D & immigration process. Set

$$P_i(z,t) = E\left(z^{N(t+s)} \mid N(s) = i\right)$$

$$= E\left(z^{N(t)} \mid N(0) = i\right)$$

Then the KFE is

$$\dot{P}_i = (\lambda z - \mu)(z-1)\frac{\partial P_i}{\partial z} + r(z-1)P_i$$

while the KBE are

$$\frac{dP_i}{dt} = (\lambda i + \nu)(P_{i+1} - P_i) + \mu i (P_{i-1} - P_i)$$

The PDE for the KFE can be solved but here the KBE is simpler. $N(t)$ is made up of a component related to the initial ancestors and a contribution from immigration (these are ind). So

$$P_i(z,t) = A(z,t)\, B(z,t)^i$$

where $A(z,t)$ is the pgf of the component derived from immigration and $B(z,t)$ that from an initial ancestor. Substitute into the KBE to get

$$\dot{A} = \nu (B-1) A \;, \quad \dot{B} = (\lambda B - \mu)(B-1)$$

& the initial conditions $(t=0)$ $A=1$, $B=z$.

$$\Rightarrow B(z,t) = \frac{\mu(1-z)e^{(\lambda-\mu)t} - (\mu - \lambda z)}{\lambda(1-z)e^{(\lambda-\mu)t} - (\mu - \lambda z)}$$

$$A(z,t) = \left[\frac{\lambda(1-z)e^{(\lambda-\mu)t} - (\mu-\lambda z)}{\lambda-\mu}\right]^{-r/\lambda}$$

If $r = 0$ then

$$P(N(t)=0 \mid N(0)=i) = B(0,i)^i$$

$$= \left(\frac{\mu e^{(\lambda-\mu)t} - \mu}{\lambda e^{(\lambda-\mu)t} - \mu}\right)^i$$