Continuous Time MC's – Part II (more details)

We are considering $\{X(t) \mid t \geq 0\}$ which is time homogeneous, Markov & has a countable state space which we take to be $\{0, 1, \cdots\}$. The pf of $X(t)$ is $\underset{\sim}{p}(t)$ & is the vector $(P(X(t)=0), P(X(t)=1), \cdots)'$. The conditional probabilities

$$p_{ij}(t) = P(X(t)=j \mid X(0)=i)$$

yield the transition function and the transition matrix

$$P(t) = \{p_{ij}(t)\}_{i,j}$$

It is easy to see

$$\underset{\sim}{p}(t)' = \underset{\sim}{p}(s)' P(t-s) \quad , \quad 0 \leq s \leq t$$

and in particular $\underset{\sim}{p}(t)' = \underset{\sim}{p}(0)' P(t)$. We also have (CKE('s))

$$P(t+s) = P(t)\, P(s) \quad \text{if } s, t \geq 0$$

Notice (1) $P(0) = I$, (2) $P(t)$ is a stochastic matrix & (3) $P(s+t) = P(s)\, P(t)$, $s, t \geq 0$. These 3 properties specify $\{P(t), t \geq 0\}$ as a <u>stochastic semigroup</u>.

Notice that $p(0)$ & $\{P(t)\}$ determine all the probabilities of the process so from a model building point of view we might consider the following steps:

1. Observe the physical phenomenon & decide if it is to be modelled as a Markov Chain.

2. If it is a Markov Chain determine (via statististical methods, intuitive reasoning, or ow) $p(0)$ & $\{P(t)\}$.

3. Use $p(0)$ & $\{P(t)\}$ to determine any probability quantities of interest for the process.

In discrete time step 2 is fairly easy as $\{P(t)\}$ is determined by $P = P(1)$, the transition matrix. In cts time step 2 is not quite so simple since there is no obvious one-step transition matrix. (no obvious "first" positive $t > 0$)

It should be noted that even non-Markov processes may have transition functions that satisfy the CKE's & form a stochastic semigroup and hence the properties of the transition function do not tell the whole story. We have already assumed that the times between transitions are independent exponential's. These have rates q_i which specify the (instantaneous) rate/intensity of leaving i. Furthermore if we observe the process only at times where changes in the state occur then we have an embedded discrete time MC with transition matrix $P=\{p_{ij}\}$. This leads to the rates/intensities

$$q_i \text{ — rate of leaving } i \text{ (in small } \Delta t)$$

$$(i \neq j) \quad q_{ij} = q_i p_{ij} \text{ — " " " to go to } j \text{ (")}$$

The effect on the transition function is seen as

$$\left.\begin{aligned} p_{ij}(\Delta t) &\approx q_{ij}\Delta t \qquad i \neq j \\ p_{ii}(\Delta t) &\approx 1 \underbrace{- q_i}_{q_{ii}} \Delta t \end{aligned}\right\} \quad (*)$$

Thus we are saying quite a bit about the behaviour of $\{P(t)\}$ at 0 & this propagates to the rest of $t > 0$ via the CKE. Essentially we are adding the assumption that

$$\underset{(G)}{(\Lambda)} = Q = P'(0^+)$$

exists. Q (or Λ or G) is called the generator of $\{P(t),\ t \geq 0\}$. Not all Q's lead to meaningful models.

If $\underbrace{\underbrace{q_i < \infty,\ \forall i}_{\text{stable case}}\ \&\ \sum_j q_{ij} = 0,\ \forall i}_{\text{conservative case}}$ then

we have

$$\dot{p}_{ij}(t) = \sum_k q_{ik}\, p_{kj}(t)$$

or in matrix form

$$\dot{P}(t) = Q\ P(t)$$

which are the Kolmogorov Backward Eq'ns.

If in addition to the conservative assumption we add the condition $\sup_i q_i < \infty$ (ie the q_i are bounded) then we have

$$\dot{p}_{ij}(t) = \sum_k p_{ik}(t)\, q_{kj}$$

or

$$\dot{P}(t) = P(t)\, Q$$

which are the forward equations. Both the BE & FE are systems of equations for an ∞ # of unknowns. There is the matter of the existance & uniqueness of solutions. Formally we have

$$P(t) = e^{Qt}$$

but one must be careful here especially for countably ∞ state spaces. For finite state space $P(t) = e^{Qt}$ is the unique solution of both the KBE & KFE.

Note The FE's solve for the dist'n of $X(t)$ given $X(0) = i$ while the BE's solve for the initial dist'n given the current state $X(t) = j$. That is the BE deals with which dist'ns lead to current states while the FE deals with where the process will be in the future given its initial state.

One consequence of (*) is that $P(t)$ is continuous (rt cts) at $t = 0$ with

$$P(t) \to P(0) = I$$

This is a reasonable assumption to make & in this case $\{P(t), t \geq 0\}$ is a standard stochastic semigroup.

In fact every standard $\{P(t)\}$ has a $Q = \dot{P}(0)$ where $q_{ij} = \dot{p}_{ij}(0)$ exists & is finite for $i \neq j$ but may be $-\infty$ for $i = j$. Write out the KFE & KBE as

$$\dot{p}_{ij}(t) = -q_j\, p_{ij}(t) + \sum_{k \neq j} p_{ik}(t)\, q_{kj} \qquad \text{(FE)}$$

$$\dot{p}_{ij}(t) = -q_i\, p_{ij}(t) + \sum_{k \neq i} q_{ik}\, p_{kj}(t) \qquad \text{(BE)}$$

We then have the following result

<u>Theorem</u> If q_i + q_{ij}, $i \neq j$ are ≥ 0 then there exists at least one solution of the BE + FE which satisfies

$$P(t+s) = P(t)P(s) \quad , s, t \geq 0$$

$$P(t) \rightarrow P(0) = I \quad \text{as } t \downarrow 0$$

$$p_{ij}(t) \geq 0 \quad , \forall i,j + t \geq 0 \ ; \ \sum_j p_{ij}(t) \leq 1 \ , \forall i + \forall t \geq 0$$

If this solution satisfies $\sum_j p_{ij}(t) = 1$, $\forall i; t \geq 0$ then it is the only sol'n of both the BE + FE.

<u>Remark</u> (a) In fact there is always a $\{p_{ij}(t)\}$ which is standard, satisfies the CKE but satisfies both the FE + BE and $\sum_j p_{ij}(t) \leq 1$ + is such that any other sol'n of either the BE or FE is $\geq p_{ij}(t)$. This is the so-called minimal solution. When it is a stochastic matrix (rows adding to 1) then it is the unique sol'n of both the FE + BE

(b) If $p_{ik}(t) > 0$ + $\sum_k p_{ik}(t) \leq 1$ then either $\sum p_{ik}(t) = 1$ $\forall t > 0$ or < 1, $\forall t > 0$

The Pure Birth Process

Here the only transitions are $j \to j+1$ with rate λ_j. We assume $\lambda_j > 0 \;\forall j$. The Q matrix is then

$$\begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & & \\ 0 & -\lambda_1 & \lambda_1 & 0 & \cdots & \\ 0 & 0 & -\lambda_2 & \lambda_2 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \end{pmatrix}$$

The FE's are then

$$\dot{P}_0(t) = -\lambda_0 P_0(t)$$

$$\dot{P}_j(t) = \lambda_{j-1} P_{j-1}(t) - \lambda_j P_j(t) \quad , j > 0$$

where $P_j(t) = p_{ij}(t)$. If we take $X(0) = 0$ then we see that these equations can be solved in succession but the probabilities may yield an improper dist'n for $X(t)$. The simplest case is $\lambda_j = \lambda$ which yields the Poisson process & then

$$P_j(t) = e^{-\lambda t} (\lambda t)^j / j!$$

which is a proper dist'n. The next simplest is $\lambda_j = \lambda j$ (here $X(0) = 1$) for which

$$P_j(t) = e^{-\lambda t} (1 - e^{-\lambda t})^{j-1}$$

This is the Yule or Yule-Furry process. If we take $X(0) = i$ then the sol'n for the dist'n of $X(t)$ is a simple convolution of geometric's which is a negative binomial. This process is also called a simple Birth process. Again we end up with a proper dist'n. For the general pure Birth process with $\lambda_j > 0$ we have

Theorem $\sum_j P_j(t) = 1 \quad \forall t \geq 0$ iff $\sum_j \frac{1}{\lambda_j} = \infty$

Proof: Write $S_m(t) = \sum_{j \leq m} P_j(t)$. Then

$$\dot{S}_m(t) = \sum_{j \leq m} \left(\lambda_{j-1} P_{j-1}(t) - \lambda_j P_j(t) \right)$$

$$= -\lambda_m P_m(t)$$

If i is the initial state $(X(0) = i)$ then $P_0(t) = 0 = P_1(t) = \cdots = P_{i-1}(t)$ & $P_i(0) = 1$. For $m \geq i$

$$1 - S_m(t) = -\int_0^t \frac{d}{du} S_m(u)\,du$$

$$= \lambda_m \int_0^t P_m(u)\,du$$

For fixed t this decreases with n, is ≥ 0 and hence has a limit. Call it $\mu(t)$.

Then

$$\mu(t) \leq \lambda_n \int_0^t P_n(u)\,du \quad , \; n \geq i$$

& so

$$\int_0^t S_n(u)\,du = \int_0^t \sum_{j \leq n} P_j(u)\,du$$

$$= \sum_{j \leq n} \int_0^t P_j(u)\,du = \sum_{i \leq j \leq n} \int_0^t P_j(u)\,du$$

$$\geq \mu(t) \sum_{i \leq j \leq n} \frac{1}{\lambda_j}$$

Now

$$\int_0^t S_n(u)\,du \leq \int_0^t 1\,du = t$$

& so

$$t \geq \mu(t) \sum_{i \leq j \leq n} \frac{1}{\lambda_j}$$

If $\sum \frac{1}{\lambda_j} = \infty$ we must then have

$$\mu(t) = 0 \quad , \; \forall t$$

& hence

$$\sum_j P_j(t) = 1 \quad , \; \forall t$$

On the other hand $\sum \frac{1}{\lambda_j} < \infty$ yields

$$\int_0^t S_n(u)du = \sum_{j=1}^{n} \int_0^t P_j(u)du$$

$$\leq \sum_{j=1}^{n} \frac{1 - S_j(t)}{\lambda_j} \qquad (P_j(u) = 0,\ j < i)$$

$$\leq \sum_{j=1}^{n} \frac{1}{\lambda_j} \leq \sum_j \frac{1}{\lambda_j}$$

Now $0 \leq S_n(u) \leq 1$ + $S_n(u)$ increases in n. Use the MCT to get

$$\int_0^t \lim_{n\to\infty} S_n(u)\,du = \lim_{n\to\infty} \int_0^t S_n(u)du \leq \sum \frac{1}{\lambda_j}$$

If $\lim_{n\to\infty} S_n(u) = 1$ this yields $t \leq \sum \frac{1}{\lambda_j}$, $\forall t$, which is impossible.

qed

<u>Remark</u> For more complex processes the situation is not so neat. Sufficient conditions for $\sum_j P_{ij}(t) = 1$ for the minimal solution do exist which hold for B + D processes (even with immigration).

For example, if the birth rate is λ_j + the death rate μ_j then $\lambda_n > 0$ for $n \geq i$ +

$$\sum_{n=i}^{\infty} \frac{\mu_i \cdots \mu_n}{\lambda_i \cdots \lambda_n} = \infty$$

$\Rightarrow$ unique sol'n of the FE + the probabilities add to 1.

Let $X \geq 0$ be integer valued. We say it is aperiodic or non-arithmetic if it is not confined to the grid $\{nd\}$ where $d > 1$. If it were then $\frac{X}{d}$ would be non-arithmetic. Another way of saying this is to require

$$\gcd \{ n \mid P(X=n) > 0 \} = 1$$

Suppose $\{ n \mid P(X=n) > 0 \} = \{ n_1, n_2, \cdots \}$.

<u>Lemma</u> $\gcd \{ n_1, n_2, \cdots \} = 1 \Rightarrow \exists$ a finite set $\{ n_1, \cdots, n_k \}$ with $\gcd = 1$.

<u>Proof</u> Let $N = \gcd \{ n_1, \cdots, n_\ell \}$. Then N is decreasing (nonincreasing) and hence has a limit. The limit must be in $\mathbb{N}$. Suppose it is i. Then $\forall \ell$ large enough $N = i$ so that i is a common factor of $\{ n_1, n_2, \cdots \}$. Hence $i = 1$ and there is a finite collection of n's with gcd 1.

<u>qed</u>

Another simple proof of this Lemma starts with the smallest $\neq 0$ element of $\{m_1, m_2, \dots\}$. Factor it into primes $p_1, \dots, p_k$ say. Now select one of the n's for which p_1 is not a factor. Select another for which p_2 isn't. This yields $k+1$ #'s with gcd 1.

<u>Lemma</u> Let $m_1, \dots, m_k \in \mathbb{N}$ be such that $\gcd\{m_1, \dots, m_k\} = 1$. Then $\exists$ integers $l_1, \dots, l_k$ such that $l_1 m_1 + \dots + l_k m_k = 1$

<u>Proof</u> Let $S = \{\underset{\sim}{n}'\underset{\sim}{m} \mid \underset{\sim}{n} \in \mathbb{Z}^k\}$ and set

$$c = \min S \cap \mathbb{N}$$

(Obviously $S \cap N$ is not empty - take $\underset{\sim}{n} = \underset{\sim}{1}$ for example.)

If $x \in S$ then $x = \overset{\in \mathbb{Z}}{q} c + r$, where $0 \leq r < c$. Suppose $c = \underset{\sim}{l}'\underset{\sim}{m}$. Then we have

$$\underset{\sim}{n}'\underset{\sim}{m} = q(\underset{\sim}{l}'\underset{\sim}{m}) + r$$

$$\Rightarrow r = (\underset{\sim}{n} - q\underset{\sim}{l})'\underset{\sim}{m} \in S.$$

Hence $r=0$ (since $r \geq 0$ & $r<c = \min S \cap \mathbb{N}$).
$\therefore$ $x = qc$ so that c divides x and each of $m_1, \dots, m_k$. But $\gcd\{m_1, \dots, m_k\}=1$ so that $c=1$

qed

<u>Corollary</u> Let $m_1, \dots, m_k \in \mathbb{N}$ have $\gcd = 1$. Then $\{\underset{\sim}{n}'\underset{\sim}{m} \mid \underset{\sim}{n} \in \mathbb{Z}^k\} = \mathbb{Z}$

<u>Remark</u> We are using the notation $\underset{\sim}{n}' = (n_1, \dots, n_k)$ and $\underset{\sim}{m}' = (m_1, \dots, m_k)$ and so on. $\mathbb{Z}^k = \{\underset{\sim}{n}\}$. Had the gcd been $d \in \mathbb{N}$ then we would have $\underset{\sim}{\ell}'\underset{\sim}{m} = d$.

Let $X \in Z^+$ be aperiodic (non-arithmetic) in the sense that X is not confined to a lattice $\{md : m = 0, 1, \cdots\}$ for integer $d > 1$. For such a rv $\exists\ k_1, \ldots, k_l$ relatively prime (ie gcd $= 1$) with $P(X = k_i) > 0$, $i = 1, \cdots, l$. It is then the case that $\exists\ N > 0$ st every $m \geq N$ can be written as a positive (≥ 0) linear combination of $k_1, \cdots, k_l$ (cf Feller, Vol I, Chapt 13, section 11 - Lemma 1). Now let $X_1, X_2, \cdots$ be iid X. If $m \geq N$ we then write it as $m_1 k_1 + \cdots + m_l k_l$, where the m's are ≥ 0. Set $M_0 = m_1 + \cdots + m_l$ (≥ 1). We then have

$$P\left(\sum_{i=1}^{M_0} X_i = m\right) \geq \prod_{i=1}^{l} \left[P(X = k_i)\right]^{m_i} > 0$$

Now, for any integer z we can find $m, n \geq N$ st $z = m - n$. Take $Y_1, Y_2, \cdots$ to be iid X & ind of the X's. We then have $N_0 \geq 1$ st

$$P\left(\sum_{j=1}^{N_0} Y_j = n\right) > 0$$

$\therefore$ for any integer z $\exists\ M_0, N_0 \geq 1$ st

$$P\left(\sum_{i=1}^{M_0} X_i - \sum_{j=1}^{N_0} Y_j = z\right) > 0 \qquad (*)$$

Proposition Let $X_1, X_2, \cdots, Y_1, Y_2, \cdots$ be ≥ 0 non-arithmetic rv's which are iid and independent of some other rv $Z \in \mathbb{Z}$ (integers). Then $\exists$ rv's $M, N > 0$ such that

$$\sum_{i=1}^{M} X_i - \sum_{j=1}^{N} Y_j \overset{wp1}{=} Z$$

Proof: Condition on $Z = z$. Since $\sum_{i=1}^{m} (X_i - Y_i)$ is a 0-mean random walk on the integers $\exists\ N_1 < N_2 < \cdots \ni \sum_{i=1}^{N_k} (X_i - Y_i) = 0$, $\forall k$ wp1.

Now let M_0, N_0 be as in our previous proposition. Set

$$\alpha = P\left(\sum_{i=1}^{N_k + M_0} X_i - \sum_{j=1}^{N_k + N_0} Y_j = z \right)$$

Then $\alpha > 0$ by our previous proposition and the definition of N_k. Notice α is the same for each k. Now take a subsequence $\{N_{k_\ell}\}$ of $\{N_k\}$ st $N_{k_{\ell+1}} - N_{k_\ell} > \max(M_0, N_0)$ and set $A_\ell = \left\{ \sum_{i=1}^{N_{k_\ell} + M_0} X_i - \sum_{j=1}^{N_{k_\ell} + N_0} Y_j = z \right\}$, $\ell = 1, 2, \cdots$

and
$$A_\ell' = \left\{ \sum_{N_{k_\ell}+1}^{N_{k_\ell}+M_0} X_i - \sum_{N_{k_\ell}+1}^{N_{k_\ell}+N_0} Y_j = z \right\}, \quad \ell = 1, 2, \cdots .$$

We have $P(A_\ell') = \alpha$ and the A_ℓ' are independent. $\therefore$ $P(A_\ell' \text{ io}) = 1$ so that $P(A_\ell \text{ io}) = 1$. Now uncondition the Z to get
$$P(A_\ell \text{ io}) = \sum_z P[\{A_\ell \text{ io}\} \mid Z = z]\, P(Z = z)$$
$$= \sum_z P(Z = z) = 1,$$
since the first part of the proof assumed $Z = z$ and actually showed $P[\{A_\ell \text{ io}\} \mid Z = z] = 1$.

qed

<u>Remark</u> This result shows that $P(T < \infty) = 1$ for the T in the coupling proof of the renewal theorem.

Uniform Integrability

Let $Y \geq 0$. Clearly

$$E(Y) < \infty \iff E[Y\, I(Y > y)] \to 0 \text{ as } y \to \infty$$

This is equivalent to the following: for any $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni$

$$P(A) < \delta \Rightarrow E(Y I_A) \leq \epsilon$$

A family of rv's $\{X_t\}$ is said to be uniformly integrable (ui) if

$$\sup_t E[|X_t|\, I(|X_t| > y)] \to 0 \text{ as } y \to \infty$$

It turns out that for $X_n \xrightarrow{P} X$ the family $\{X_n\}$ is ui $\iff$ $X_n, X \in L_1$ & $X_n \xrightarrow{L_1} X$

__Remark__ $X \in L_1$ means $E(|X|) < \infty$. $X_n \xrightarrow{L_1} X$ means $X_n \in L_1$ & $E(|X_n - X|) \to 0$. Then $E(X_n) \to E(X)$.

A useful lemma in showing these results is $\{X_n\}$ is ui iff $\sup_n E|X_n| < \infty$ and for all $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni$

$$P(A) < \delta \Rightarrow \sup_n E(|X_n| I_A) \leq \epsilon$$