Martingales (discrete time)

Let $X_1, X_2, \ldots$ satisfy $E(X_n / \underset{\sim}{X}_{n-1}) = 0$, $\forall n \geq 1$ (we take X_0 to be a constant) & set

$$S_n = \sum_{k=1}^{n} X_k \qquad (S_0 = 0)$$

$\{S_n\}$ is termed a 0-mean martingale. For S_0 a constant (not necessarily 0)

$$S_n = S_0 + \sum_{k=1}^{n} X_k$$

is called a martingale. Notice $E(S_n) = S_0$, $\forall n$ in this case. It is easily seen that

$$E(S_{n+1} / \underset{\sim}{S}_n) = S_n$$

which is usually taken as the defining property of a martingale. From here we see

$$E(S_n / \underset{\sim}{S}_m) = S_m \quad , \quad m < n$$

& so $E(S_n - S_m / \underset{\sim}{S}_m) = 0$. The X's are called martingale differences & there is a 1-1 relationship between $\underset{\sim}{X}_n$ & $\underset{\sim}{S}_n$. One consequence of this is

$$E(S_n / \underset{\sim}{X}_m) = S_m \quad \text{or} \quad E(S_n - S_m / \underset{\sim}{X}_m) = 0,$$

for $m < n$.

If $\{Y_n\}$ is such that $X_n | S_n$ is a f'n of $\underset{\sim}{Y}_n$ & $E(S_{n+1} | \underset{\sim}{Y}_n) = S_n$ then of course $E(S_{n+1} | \underset{\sim}{S}_n) = E[E(S_{n+1} | \underset{\sim}{Y}_n) | \underset{\sim}{S}_n]$

$$= E(S_n | \underset{\sim}{S}_n) = S_n$$

so that $\{S_n\}$ is a martingale. We often say, in this case, that $\{S_n\}$ is a martingale wrt $\{Y_n\}$.

eg. Let $\{S_n\}$ be a MC so that

$$E[g(S_{n+1}) | \underset{\sim}{S}_n] = E[g(S_{n+1}) | S_n]$$

Suppose we can find ψ (psi - lapse) $\ni$

$$E[\psi(S_{n+1}) | S_n] = \psi(S_n)$$

which is $P\underset{\sim}{\psi} = \underset{\sim}{\psi}$, where P is the transition matrix. We would then have

$$E[\psi(S_{n+1}) | \underset{\sim}{S}_n] = \psi(S_n)$$

& hence $\{\psi(S_n)\}$ is a martingale (wrt $\{S_n\}$).

We then have $E[\psi(S_n)]$ is constant $\forall n$ which suggests that, for random T, $E[\psi(S_T)]$ will also be this constant. Assume for now this to be the case ~~for now~~ (it isn't always, but is for stopping times – optional stopping theorem). Consider the simple random walk on $\{0, 1, \ldots, N-1, N\}$ where $S_0 = k \in \{1, \ldots, N-1\}$, unit steps + absorbing states $0, N$. We wish to calculate the probability that the chain stops at 0. As before $\{S_n\}$ is a MC & we let p be the prob of a $+1$ step & $q = 1-p$ the prob of a step to the left. We assume $0<p<1$. Let

$$\psi(S_n) = (q/p)^{S_n}$$

It is then easily seen $E[\psi(S_{n+1}) \mid \underset{\sim}{X}_n] = \psi(S_n)$ where $X_1, X_2, \ldots$ are the steps. We then have

$$E[\psi(S_n)] = E[\psi(S_0)] = (q/p)^k \quad , \forall n$$

Let T = time to absorbtion (into 0 or N).

It would appear reasonable that

$$E[\psi(S_T)] = (q/p)^k \Rightarrow E\left[(q/p)^{S_T}\right] = (q/p)^k$$

Now $S_T = 0$ or N. Let $q_k = P(S_T = 0 \mid S_0 = k)$ & $p_k = 1 - q_k$.

Then $$E\left[(q/p)^{S_T}\right] = q_k + \left(\frac{q}{p}\right)^N p_k = \left(\frac{q}{p}\right)^k$$

Set $\rho = q/p$ & assume $\rho \neq 1$. We then get $q_k = \dfrac{\rho^k - \rho^N}{1 - \rho^N}$

Now $\{S_n\}$ is a martingale (MG) if $E(S_{n+1} | \underset{\sim}{S}_n) = S_n$.
If = is replaced by $\geq$ then we have a submartingale (SMG). If by $\leq$ then we have a supermartingale. I'll leave it to you to show

Problem (a) $\{S_n\}$ a MG & g convex $\Rightarrow \{g(S_n)\}$ is a SMG.
(b) $\{S_n\}$ a SMG, g convex & inc $\Rightarrow \{g(S_n)\}$ a SMG.
(c) $\{S_n\}$ a SMG $\Rightarrow \{S_n^+\}$ a SMG

Note - $X^+ = \max(0, X)$. Another notation used is $0 \overset{\max}{\vee} X$
- $X \wedge Y$ denotes $\min(X, Y)$ & $X \vee Y$ the max.

If $\{S_n\}$ is a martingale wrt $\{Y_n\}$ then

$$S_n = E(S_{n+N} | \underset{\sim}{Y}_n)$$

If S_{n+N} converged to S_∞, with $E|S_\infty| < \infty$, we might then conclude

$$S_n = E(S_\infty | \underset{\sim}{Y}_n) \qquad (*)$$

If we start with S_∞ having finite mean & define S_n via $(*)$ then $\{S_n\}$ is a martingale (the Doob martingale).

If we are willing to assume 2nd moments (*) may be interpreted in a least-squares prediction context.

<u>Kolmogorov's Inequality</u> Let $\{S_n\}$ be a 0-mean martingale & $c > 0$. Then

$$P\left(\max_{1\le k\le n} |S_k| > c\right) \le \frac{Var(S_n)}{c^2} = \frac{E(S_n^2)}{c^2}$$

<u>Proof</u>: Let $M_j = \max_{1\le k\le j} |S_k|$ & $A_j = \{M_{j-1} \le c < M_j\}$, $j=1,\cdots,n$ $(M_0 = 0)$. Then $\{\max_{1\le k\le n} |S_k| > c\} = \bigcup_{j=1}^{n} A_j$ & the A's are disjoint. Now,

$$\begin{aligned} E(S_n^2 I_{A_j}) &= E\left[(S_n - S_j + S_j)^2 I_{A_j}\right] \\ &= E(S_j^2 I_{A_j}) + E\left[(S_n - S_j)^2 I_{A_j}\right] \\ &\qquad + 2\underbrace{E(S_j(S_n - S_j) I_{A_j})}_{0} \\ &\ge E(S_j^2 I_{A_j}) \end{aligned}$$

since $E\left[(S_n - S_j)^2 I_{A_j}\right] \ge 0$ & $E\left[(S_n - S_j) f^j_n(\underset{\sim}{X}_j)\right] = 0$.

$\therefore$ $E(S_n^2 I_{A_j}) \ge E(S_j^2 I_{A_j}) \ge E(c^2 I_{A_j}) = c^2 P(A_j)$, since $A_j \Rightarrow |S_j| > c$ $\therefore$ $E(S_n^2) \ge \sum_{j=1}^{n} E(S_n^2 I_{A_j}) \ge c^2 \sum_{j=1}^{n} P(A_j)$ & so $P(\max_{1\le k\le n} |S_k| > c) \le E(S_n^2)/c^2$ qed

This inequality in the iid case leads to a proof of the SLLN. We can extend the SLLN to the dependent case quite easily with a generalization of the inequality (due to Hajek & Renyi)

<u>Theorem (The KHR Inequality)</u> Let $\{S_n\}$ be a 0-mean martingale & $0 = c_0 < c_1 \le \cdots$ constants. Then

$$P(|S_k| \le c_k \ , k = 1, \cdots, n) \geq 1 - \sum_{k=1}^{n} \frac{E(X_k^2)}{c_k^2}$$

<u>Remark</u> Note $S_n = \sum_{k=1}^{n} X_k$ & the X's are uncorrelated with $E(X_k) = 0$. If $c_k = c > 0$ for $k > 0$ this result reduces to the Kolmogorov Inequality as $Var(S_n) = E(S_n^2) = \sum_{k=1}^{n} E(X_k^2)$

<u>Proof</u>

Let $B_n = \{ |S_1| \le c_1, \cdots, |S_n| \le c_n \}$. Then

$$\begin{aligned}
P(B_n) &= E[I(B_n)] \\
&= E[I(B_{n-1}) I(|S_n| \le c_n)] \\
&= E[I(B_{n-1})(1 - I(|S_n| > c_n))] \\
&\geq E[I(B_{n-1})(1 - S_n^2/c_n^2)] \\
&= E[I(B_{n-1})(1 - (S_{n-1} + X_n)^2/c_n^2)] \\
&= E[I(B_{n-1})(1 - S_{n-1}^2/c_n^2 - X_n^2/c_n^2)] ,
\end{aligned}$$

since $E(X_n S_{n-1} I(B_{n-1})) = 0$.

$\therefore \quad P(B_m) \geq E\left[I(B_{m-1})\left(1-\frac{S_{m-1}^2}{c_m^2}\right)\right] - E\left[I(B_{m-1})X_m^2/c_m^2\right]$

$$\geq E\left[I(B_{m-1})\left(1-\frac{S_{m-1}^2}{c_{m-1}^2}\right)\right] - E\left(\frac{X_m^2}{c_m^2}\right)$$

$\left(\because \; c_m^2 \geq c_{m-1}^2 \;\; \& \;\; \frac{X_m^2}{c_m^2} \geq I(B_{m-1})\frac{X_m^2}{c_m^2}\right)$

Now $I(B_{m-1}) = I(B_{m-2})\, I(|S_{m-1}| \leq c_{m-1})$. Since $I(|S_{m-1}| \leq c_{m-1})\left(1-\frac{S_{m-1}^2}{c_{m-1}^2}\right) \geq 1-\frac{S_{m-1}^2}{c_{m-1}^2}$ we have

$$P(B_m) \geq E\left[I(B_{m-1})(1-S_m^2/c_m^2)\right] \qquad (*)$$

$$\geq E\left[I(B_{m-2})\left(1-\frac{S_{m-1}^2}{c_{m-1}^2}\right)\right] - E\left(\frac{X_m^2}{c_m^2}\right) \qquad (**)$$

Now apply the reduction from (*) to (**) to the 1st term of (**) repeatedly to finally obtain (set $B_0 = \Omega$)

$$P(B_m) \geq 1 - \sum_{k=1}^{m} \frac{E(X_m^2)}{c_m^2}$$

qed

<u>Note</u> When $m = 2$ & going from (*) to (**) the 1st term in (**) will be $1 - \frac{E(S_1^2)}{c_1^2} = 1 - \frac{E(X_1^2)}{c_1^2}$

Theorem Let $\{X_n\}$ have constant finite variance σ^2 and satisfy $E(X_n \mid \underset{\sim}{X}_{n-1}) = 0$. Then

$$\bar{X}_n = \frac{S_n}{n} \xrightarrow{as} 0$$

Proof For any $N > 0$ the sequence $0, S_N, \underbrace{S_N + X_{N+1}}_{S_{N+1}}, \cdots$ is a 0-mean martingale. Now let $\epsilon > 0$ & consider

$$P\left(\left|\frac{S_n}{n}\right| \le \epsilon, \ \forall n \geq N_0 \right)$$

$$= P\left(|S_{N_0}| \le N_0 \epsilon, |S_{N_0+1}| \le (N_0+1)\epsilon, \cdots \right)$$

$E(X^2_{N_0+1})$, etc....

$$\overset{KHR}{\geq} 1 - \left[\frac{E(S^2_{N_0})}{\epsilon^2 N_0^2} + \sum_{k=N_0+1}^{\infty} \frac{\sigma^2}{\epsilon^2 k^2} \right]$$

Since $\dfrac{E(S^2_{N_0})}{\epsilon^2 N_0^2} = \dfrac{N_0 \sigma^2}{\epsilon^2 N_0^2} \to 0$ as $N_0 \to \infty$ as does $\displaystyle\sum_{k=N_0+1}^{\infty} \frac{\sigma^2}{\epsilon^2 k^2}$ we conclude $\bar{X}_n \xrightarrow{as} 0$

qed

Note $\dfrac{S_n}{a_n} \xrightarrow{as} 0$ for any $a_n > 0$ with $\sum \frac{1}{a_n^2} < \infty$.

The Martingale Convergence Theorem

Let $\{S_n\}$ be a 0-mean martingale with $\sup_n E(S_n^2) < \infty$. Then $\exists$ a rv $S_\infty \in L_2$ st $S_n \xrightarrow[ms]{as} S_\infty$.

Remark (a) The 0-mean assumption can easily be relaxed as any nonzero mean martingale is a constant + a 0-mean one. More importantly the 2nd moment condition can be weakened to $\sup_n E|S_n| < \infty$ & then the conclusion is $S_n \xrightarrow{as} S_\infty$ with $E(|S_\infty|) < \infty$. If $\{S_n\}$ is ui we also have $S_n \xrightarrow{L_1} S_\infty$.

(b) The remarks in (a) continue to apply if $\{S_n\}$ is a SMG.

Proof (the L_2 version)

$$\sup E(S_n^2) < \infty \Rightarrow \sum_{k=1}^{\infty} E(X_k^2) < \infty$$

$$\Rightarrow \sum_{k=1}^{n} X_k \xrightarrow{ms} S_\infty$$

with $E(S_\infty^2) < \infty$. It remains to show that $\sum_{k=1}^{n} X_k \xrightarrow{as}$. If it does then the limit must be S_∞.
($\because$ a subseq of $\sum_{k=1}^{n} X_k \xrightarrow{as} S_\infty$)

Now for every n the sequence $0, S_{n+1}-S_n, S_{n+2}-S_n, \ldots$ is a 0-mean martingale. Now apply Kolmogorov's Inequality to get

$$P(|S_m - S_n| \le \epsilon \ , \forall m > n) \geq 1 - \frac{1}{\epsilon^2} \sum_{n+1}^{\infty} E(X_k^2)$$

$$\rightarrow 1$$

Hence $\{S_n\}$ is as mutually convergent (ie has the Cauchy property) & so $S_n \xrightarrow{as} (\ )$ as $n \to \infty$. The limit must be S_∞ (wp1).

<u>qed</u>

<u>The Optional Stopping Theorem</u>

Let $\{S_n\}$ be a martingale with mean S_0. If T is a stopping time for $\{S_n\}$ and

(a) $T \overset{as}{<} \infty$ (b) $E(|S_T|) < \infty$ (c) $E[S_n I(T \geq n)] \to 0$

then $E(S_T) = E(S_n) = S_0$.

<u>Remarks</u>

- T is a stopping time if $\forall n$ $\{T \le n\}$ is a $\underset{\sim}{S}_n$ event
- If (a) & (c) hold & $\sup_n E|S_n| < \infty$ then (b) holds
- If $1 \le T_1 \le T_2 \le \cdots$ are stopping times then $\{S_{T_k}\}$ is a MG

– the result holds if $E(T) < \infty$ & $\exists$ a constant M st

$$\sup_n E(|X_{n+1}| \mid \underset{\sim}{X}_n) \le M$$

This can be further weakened by only taking the sup over $n \le T$.

__Proof__ For $n > j$

$$E[S_n I(T=j)] = E[(S_n - S_j) I(T=j)] + E[S_j I(T=j)]$$
$$= E[S_j I(T=j)]$$

$\therefore$ for any n

$$S_0 = E(S_n) = E[S_n I(T \ge n)] + E[S_n I(T < n)]$$
$$= E[S_n I(T \ge n)] + \sum_{j=1}^{n-1} E[S_n I(T=j)]$$
$$= E[S_n I(T \ge n)] + \sum_{j=1}^{n-1} E[S_j I(T=j)]$$

But $$E[S_T I(T < n)] = \sum_{j=1}^{n-1} E[S_T I(T=j)] = \sum_{j=1}^{n-1} E[S_j I(T=j)]$$

so that

$$E(S_T) - S_0 = E[S_T I(T \ge n)] - E[S_n I(T \ge n)]$$

Now, $E(S_n I(T \ge n)) \to 0$ by assumption and

$$|E(S_T I(T \geq n))|$$

$$\leq E(|S_T| I(T \geq n)) = \sum_{j=n}^{\infty} E(|S_T| I(T=j))$$

$$= \sum_{j=n}^{\infty} E(|S_j| I(T=j)) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

since

$$\infty > E(|S_T|) = E\Big(|S_T| \sum_{j=1}^{\infty} I(T=j)\Big)$$

$$= \sum_{j=1}^{\infty} E(|S_T| I(T=j))$$

$$= \sum_{j=1}^{\infty} E(S_j I(T=j))$$

Hence

$$E(S_T) = S_0$$

<u>qed</u>

Theorem Let $\{S_n\}$ be a martingale, $T \geq 1$ a stopping time & $Z_n = S_{T \wedge n}$. Then $\{Z_n\}$ is a martingale.

Proof $Z_n = Z_n I(T<n) + Z_n I(T \geq n)$

$$= \sum_{j=1}^{n-1} Z_n I(T=j) + S_n I(T \geq n)$$

$$= \sum_{j=1}^{n-1} S_j I(T=j) + S_n I(T \geq n)$$

which is a f'n of $\underset{\sim}{S}_n$. Now

$$E(Z_{n+1} | \underset{\sim}{S}_n) = \sum_{j=1}^{n} S_j I(T=j) + E(S_{n+1} I(T \geq n+1) | \underset{\sim}{S}_n)$$

Since $\{T \geq n+1\} = \{T \leq n\}^c$ is an $\underset{\sim}{S}_n$-event we get

$$E(S_{n+1} I(T \geq n+1) | \underset{\sim}{S}_n) = I(T \geq n+1) E(S_{n+1} | \underset{\sim}{S}_n)$$
$$= I(T > n) S_n$$

$$\therefore E(Z_{n+1} | \underset{\sim}{S}_n) = S_n I(T>n) + Z_n I(T \leq n)$$
$$= Z_n I(T>n) + Z_n I(T \leq n)$$
$$= Z_n$$

qed