Uniform Integrability

Let $Y \geq 0$. Clearly

$$E(Y) < \infty \iff E[Y\,I(Y>y)] \to 0 \text{ as } y \to \infty$$

This is equivalent to the following: for any $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni$

$$P(A) < \delta \Rightarrow E(Y I_A) \leq \epsilon$$

A family of rv's $\{X_t\}$ is said to be uniformly integrable (ui) if

$$\sup_t E[|X_t|\,I(|X_t|>y)] \to 0 \text{ as } y \to \infty$$

It turns out that for $X_n \xrightarrow{P} X$ the family $\{X_n\}$ is ui $\iff$ $X_n, X \in L_1$ & $X_n \xrightarrow{L_1} X$

<u>Remark</u> $X \in L_1$ means $E(|X|) < \infty$. $X_n \xrightarrow{L_1} X$ means $X_n \in L_1$ & $E(|X_n - X|) \to 0$. Then $E(X_n) \to E(X)$.

A useful lemma in showing these results is $\{X_n\}$ is ui iff $\sup_n E|X_n| < \infty$ and for all $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni$

$$P(A) < \delta \Rightarrow \sup_n E(|X_n| I_A) \leq \epsilon$$