The next weeks

March 2	§10.5 Count data and log-linear models
March 9	§10.6 Overdispersion and quasi-likelihood, GEEs
March 16	§10.7 Semiparametric models
March 23	Generalized additive models and lasso
March 30	Finishing pieces, + review

Homework 3: due April 2, 5 pm

Final Test: April 17, 1 - 3 pm

Fitting Poisson models

- suppose we have 3 factors, each with several levels
- observe a response at each combination of factors
- linear model might be

$$y_{ijk} = \mu+\alpha_i+\beta_j+\gamma_k+\epsilon_{ijk}, \quad k = 1,\ldots,K; j = 1,\ldots,J; i = 1,\ldots I$$

- or

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + \epsilon_{ijk}$$

- if the y_{ijk} are positive counts, rather than continuous, then Poisson model could have

$$y_{ijk} \sim Po(\mu_{ijk}), \quad \log(\mu_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_k$$

- or

$$\log(\mu_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk}$$

Example 10.22: Jacamar data

	Aphrissa boisduvalli N/S/E	*Phoebis argante* N/S/E	*Dryas iulia* N/S/E	*Pierella luna* N/S/E	*Consul fabius* N/S/E	*Siproeta stelenes*† N/S/E
Unpainted	0/0/14	6/1/0	1/0/2	4/1/5	0/0/0	0/0/1
Brown	7/1/2	2/1/0	1/0/1	2/2/4	0/0/3	0/0/1
Yellow	7/2/1	4/0/2	5/0/1	2/0/5	0/0/1	0/0/3
Blue	6/0/0	0/0/0	0/0/1	4/0/3	0/0/1	0/1/1
Green	3/0/1	1/1/0	5/0/0	6/0/2	0/0/1	0/0/3
Red	4/0/0	0/0/0	6/0/0	4/0/2	0/0/1	3/0/1
Orange	4/2/0	6/0/0	4/1/1	7/0/1	0/0/2	1/1/1
Black	4/0/0	0/0/0	1/0/1	4/2/2	7/1/0	0/1/0

† includes *Philaethria dido* also.

Table 10.2 Response of a rufous-tailed jacamar to individuals of seven species of palatable butterflies with artifically coloured wing undersides. (N=not sampled, S = sampled and rejected, E = eaten)

- $c = 1, \ldots, 8; s = 1, \ldots, 6; f = 1, 2, 3$
- $y_{csf} \sim Po(\mu_{csf})$; $\log \mu_{csf} = \mu + \alpha_c + \beta_s + (\alpha\beta)_{cs} + \gamma_f + (\alpha\gamma)_{cf} + (\beta\gamma)_{sf}$
- $y_{csf} \mid y_{cs+} \sim Trinomial(y_{cs+}; \pi)$

... Example 10.22

- $y_{csf} \sim Po(\mu_{csf})$
  $\log \mu_{csf} = \mu + \alpha_c + \beta_s + \gamma_f + (\alpha\beta)_{cs} + (\alpha\gamma)_{cf} + (\beta\gamma)_{sf}$

  ← incl. terms for ~~fixed~~ covariates

- $y_{csf} \mid y_{cs+} \sim Trinomial(y_{cs+}; \frac{\mu_{csf}}{\mu_{cs+}})$

  $\pi_{cs1} = \frac{\mu_{cs1}}{\mu_{cs+}}$

- $y_{cs+} \sim Po(\mu_{cs+})$

  $\mu_{cs+} =$ ~~N~~ $\mu_{cs1} + \mu_{cs2} + \mu_{cs3}$

- want to assess effect of colour and species on fate; e.g. if $(\alpha\gamma)_{cf} \equiv 0$,

  $p(\text{Fate } f)$ ind't of colour

- all models need to include $\alpha_c + \beta_s + (\alpha\beta)_{cs}$ – why?

this fixes the total in cell (c,s)

[p. 501 ← γ2 ← αcs]

$$\mu_{csf} = e^{\mu + \alpha_c + \beta_s + \gamma_f + (\alpha\beta)_{cs} + (\alpha\gamma)_{cf} + (\beta\gamma)_{sf}}$$

$$\pi_{\underline{csf}} = \mu_{csf} / (\mu_{cs1} + \mu_{cs2} + \mu_{cs3})$$

$$= \frac{e^{\gamma_f + (\alpha\gamma)_{cf} + (\beta\gamma)_{sf}}}{\sum_{f=1}^{3} e^{\gamma_f + (\alpha\gamma)_{cf} + (\beta\gamma)_{sf}}}$$

simpler: $e^{\gamma_f} / \sum e^{\gamma_f}$ no eff. of color sp

... Example 10.22

y col sp N S E
Ab 0 0 14
. . . 6 1 0
jacamar

```
> jac.long[1:6,]
  jac.y   jac.col jac.sp jac.fate
1     0 Unpainted     Ab        N
2     0 Unpainted     Ab        S
3    14 Unpainted     Ab        E
4     6 Unpainted     Pa        N
5     1 Unpainted     Pa        S
6     0 Unpainted     Pa        E
...
> jac.glm0 = glm(jac.y ~ jac.col*jac.sp + jac.fate, data = jac.long, family = poisson)
> summary(jac.glm0)
...
    Null deviance: 385.55  on 143  degrees of freedom
Residual deviance: 173.86  on  94  degrees of freedom
AIC: 476.99

Number of Fisher Scoring iterations: 15
> jac.glm1 = update(jac.glm0, .~. + jac.fate:(jac.col+jac.sp))
Warning message:
glm.fit: fitted rates numerically 0 occurred
> summary(jac.glm0)
...
    Null deviance: 385.554  on 143  degrees of freedom
Residual deviance:  90.659  on  70  degrees of freedom
AIC: 441.79

Number of Fisher Scoring iterations: 17
```

Example 10.23

$$\mu_{td} = T_{td}\left(\beta_0 + \beta_1 d^{\beta_2}\right) t^{\beta_3}$$

Years of smoking t	Daily cigarette consumption d						
	Nonsmokers	1–9	10–14	15–19	20–24	25–34	35+
15–19	10366/1	3121	3577	4317	5683	3042	670
20–24	8162	2937	3286/1	4214	6385/1	4050/1	1166
25–29	5969	2288	2546/1	3185	5483/1	4290/4	1482
30–34	4496	2015	2219/2	2560/4	4687/6	4268/9	1580/4
35–39	3512	1648/1	1826	1893	3646/5	3529/9	1336/6
40–44	2201	1310/2	1386/1	1334/2	2411/12	2424/11	924/10
45–49	1421	927	988/2	849/2	1567/9	1409/10	556/7
50–54	1121	710/3	684/4	470/2	857/7	663/5	255/4
55–59	826/2	606	449/3	280/5	416/7	284/3	104/1

Table 1.4 Lung cancer deaths in British male physicians (Frome, 1983). The table gives man-years at risk/number of cases of lung cancer, cross-classified by years of smoking, taken to be age minus 20 years, and number of cigarettes smoked per day.

```
> data(lung.cancer)
> lung.cancer[1:3,]
  years.smok cigarettes  Time y
1      15-19          0 10366 1
2      15-19        1-9  3121 0
3      15-19      10-14  3577 0
```

$$y_{td} \sim Po(T_{td} \cdot \mu_{td})$$

T # man-years

$$\mu_{td} = e^{\alpha_t + \beta_d}$$ no χ^2

... Example 10.23

$E(Y_{td}) = T_{td} \cdot e^{\alpha_t + \beta_d}$

```
> summary(glm(y ~ cigarettes + years.smok + offset(log(Time)),
family = poisson, data = lung.cancer))
...

Coefficients:
                 Estimate Std. Error z value Pr(>|z|)
(Intercept)      -12.5784     1.1475 -10.961  < 2e-16 ***
cigarettes1-9      1.2200     0.7073   1.725 0.084547 .
cigarettes10-14    2.0991     0.6363   3.299 0.000971 ***
cigarettes15-19    2.3089     0.6327   3.649 0.000263 ***
cigarettes20-24    2.9009     0.5956   4.870 1.11e-06 ***
cigarettes25-34    3.1162     0.5947   5.240 1.61e-07 ***
cigarettes35+      3.6059     0.6048   5.962 2.49e-09 ***
...
(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 445.099  on 62  degrees of freedom
Residual deviance:  51.471  on 48  degrees of freedom
AIC: 201.31
```

$e^{3.6059 \pm 2(.6048)}$

... log-linear models

- see also Example 10.21 for a Poisson example (y is number of goals scored in soccer match)
- with the Poisson-multinomial connection, we can also fit contingency tables with more than one response factor
- example HW 2 Q4: treat wheeze and breathlessness as responses, age as covariate
- reference Faraway: Extending the Linear Model with R; Agresti: Analysis of Categorical Data
- skip "marginal models" (p.505) and ordinal data (§10.5.2)

as.vector(t(as.matrix(jacamar[,3:5])))

Over-dispersion §10.6

- over-dispersion means Var(Y) is larger than expected under the Poisson or Binomial model
- which specify Var(Y) $= \mu$, or $v(\mu) = \mu(1-\mu)/m$
- where does over-dispersion come from? possibly multiplicative "noise", see p. 511 for Poisson, (10.34) for Binomial
- likelihood analysis computes marginal density, averaged over noise – e.g. Poisson $\rightarrow$ Negative Binomial (Ex. 10.26)
- alternative analysis based on "quasi-likelihood" uses analogy with least squares
- recall that if $E(Y) = X\beta$, Var(Y) $= \sigma^2 I$, then $\hat{\beta}$ is best linear unbiased estimator of β, even if Y is not normally distributed (Gauss-Markov theorem)
- there could be better nonlinear estimators of β

... overdispersion

- if $E(Y) = X\beta$ and $\text{Var}(Y) = V$, then $\hat{\beta} =$ unbiased for β
- $\text{Var}(\hat{\beta}) =$ (8.19)
- if we knew V, replace $\hat{\beta}$ by weighted least squares estimator; otherwise, use $\hat{\beta}$ and adjust confidence intervals by some estimate of V, see p.377

... overdispersion

- estimation of β in a generalized linear model depends only on the specification of the mean function
- and the variance function
- suggests using the same estimating equation for β, but allow inflation of the variance function by an unknown dispersion parameter
- e.g. $E(y_j) = \mu_j, \quad \text{Var}(y_j) = \phi\mu_j$ – overdispersed Poisson
- e.g. $E(y_j) = \mu_j, \quad \text{Var}(y_j) = \phi\pi_j(1-\pi_j)/m$ – overdispersed Binom
- estimating equation for β is unchanged

... overdispersion

- $$\sum_{j=1}^{n} x_j \frac{y_j - \mu_j}{g'(\mu_j) V(\mu_j)} = 0$$
- this is an unbiased estimating function $g(y; \beta)$; satisfies $E\{g(Y; \beta)\} = 0$
- under some regularity conditions the solution of $g(y; \tilde{\beta}) = 0$ is consistent, asymptotically normal
- a. $\text{Var}(\tilde{\beta}) = \phi(X^T W X)^{-1}$
- from general theory on unbiased estimating functions

$$E\left\{-\frac{\partial g(Y;\beta)}{\partial \beta}\right\}^{-1} Var\{g(Y;\beta)\} E\left\{-\frac{\partial g(Y;\beta)}{\partial \beta}\right\}^{-1}$$

... inference

- $\hat{\phi} = \frac{1}{n-p} \sum_{j=1}^{n} \sum_{j=1}^{n} \frac{(y_j - \hat{\mu}_j)^2}{V(\hat{\mu}_j)}$

Pearson's χ^2 / $n-p$

- $\widehat{\text{Var}}(\tilde{\beta}_j) = \hat{\phi}\, \widehat{\text{Var}}(\hat{\beta}_j)$

from glm

- comparison of models: $A \subset B$ $D_A - D_B \dot{\sim} \chi^2_{p_B - p_A}$
- changes to

$$\frac{(D_A - D_B)/(p_B - p_A)}{\hat{\phi}} \sim F_{p_B - p_A,\, p_B}$$

from model B

Example 10.28

Set 2

Table 2 (Bissell, 1972) gives the numbers of faults in rolls of textile fabric. The distribution of number of faults is of interest, especially in its relation to that expected if faults occur at random at a fixed rate per metre.

Table 2. Numbers of faults in rolls of textile fabric

Roll No.	Roll length (metres)	No. of faults	Roll No.	Roll length (metres)	No. of faults
1	551	6	17	543	8
2	651	4	18	842	9
3	832	17	19	905	23
4	375	9	20	542	9
5	715	14	21	522	6
6	868	8	22	122	1
7	271	5	23	657	9
8	630	7	24	170	4
9	491	7	25	738	9
10	372	7	26	371	14
11	645	6	27	735	17
12	441	8	28	749	10
13	895	28	29	495	7
14	458	4	30	716	3
15	642	10	31	952	9
16	492	4	32	417	2

Cox & Snell

A. Stat.

... example 10.28

$$E\,Y_j = \beta x_j$$

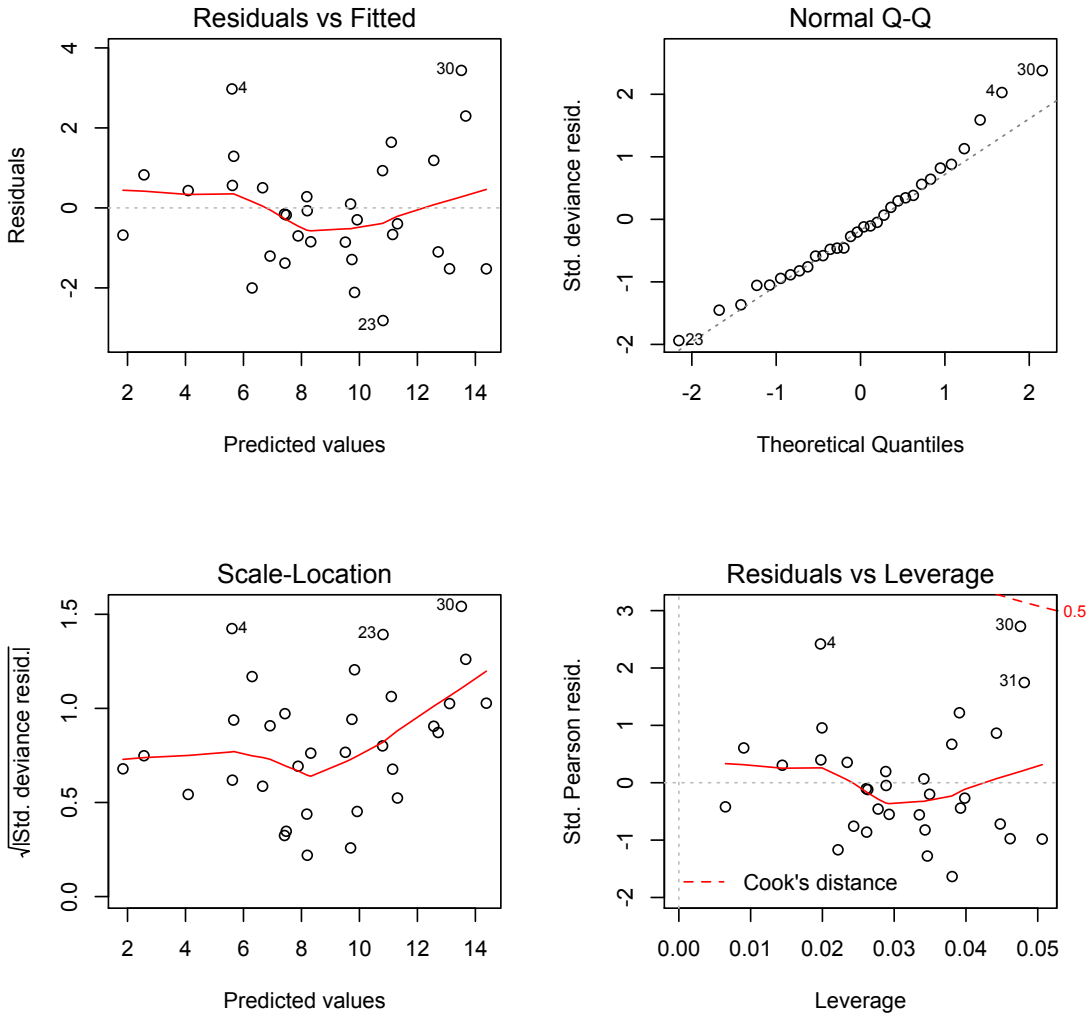
```
> data(cloth)
> cloth[1:5,]
     x  y
1 1.22  1
2 1.70  4
3 2.71  5
4 3.71 14
5 3.72  7
> with(cloth,plot(x,y)) # gives Fig 10.11
> cloth.glm0 = glm(y ~ x - 1, family = poisson(link = identity), data = cloth)
> summary(cloth.glm0)
Coefficients:
  Estimate Std. Error z value Pr(>|z|)
x  1.51024    0.08962   16.85   <2e-16 ***
---

(Dispersion parameter for poisson family taken to be 1)

    Null deviance:    Inf  on 32  degrees of freedom
Residual deviance: 64.537  on 31  degrees of freedom
> cloth.glm1 = glm(y ~ x - 1, family = quasipoisson(link = identity), data = cloth)
> summary(cloth.glm1)
Coefficients:
  Estimate Std. Error t value Pr(>|t|)
x   1.5102     0.1328   11.38 1.35e-12 ***
---

(Dispersion parameter for quasipoisson family taken to be 2.194371)
```

$$\hat{\phi} = X^2/(n-1)$$

$$\widehat{\mathrm{var}}\,\tilde{\beta} = \hat{\phi} \times \widehat{\mathrm{var}}\,\hat{\beta}$$

negative binomial p. 515

Quasi-Poisson model fit



Example 10.29

$$\log \frac{p}{1-p} \sim \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$$

City	Rain	r/m	City	Rain	r/m	City	Rain	r/m	City	Rain	r/m
1	1735	2/4	11	2050	7/24	21	1756	2/12	31	1780	8/13
2	1936	3/10	12	1830	0/1	22	1650	0/1	32	1900	3/10
3	2000	1/5	13	1650	15/30	23	2250	8/11	33	1976	1/6
4	1973	3/10	14	2200	4/22	24	1796	41/77	34	2292	23/37
5	1750	2/2	15	2000	0/1	25	1890	24/51			
6	1800	3/5	16	1770	6/11	26	1871	7/16			
7	1750	2/8	17	1920	0/1	27	2063	46/82			
8	2077	7/19	18	1770	33/54	28	2100	9/13			
9	1920	3/6	19	2240	4/9	29	1918	23/43			
10	1800	8/10	20	1620	5/18	30	1834	53/75			

Table 10.19 Toxoplamosis data: rainfall (mm) and the numbers of people testing positive for toxoplasmosis, r, our of m people tested, for 34 cities in El Salvador (Efron, 1986).

Terms	df	Deviance
Constant	33	74.21
Linear	32	74.09
Quadratic	31	74.09
Cubic	30	62.63

Table 10.20 Analysis of deviance for polynomial logistic models fitted to the toxoplasmosis data.

- incidence of toxoplasmosis as a function of rainfall
- residual deviances approximately twice the degrees of freedom

... example 10.29

```
> data(toxo)
> toxo[1:4,]
  rain  m  r
1 1620 18  5
2 1650 30 15
3 1650  1  0
4 1735  4  2
> toxo.glm0 = glm(cbind(r,m-r) ~ rain + I(rain^2) + I(rain^3), data = toxo,
family = binomial)
> anova(toxo.glm0)
 ...
          Df Deviance Resid. Df Resid. Dev
NULL                         33     74.212
rain       1   0.1244        32     74.087
I(rain^2)  1   0.0000        31     74.087
I(rain^3)  1  11.4529        30     62.635
> toxo.glm1 = glm(cbind(r,m-r) ~ rain + I(rain^2) + I(rain^3), data = toxo,
family = quasibinomial)
> summary(toxo.glm1)
...
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -2.902e+02  1.215e+02  -2.388   0.0234 *
rain         4.500e-01  1.876e-01   2.398   0.0229 *
I(rain^2)   -2.311e-04  9.616e-05  -2.404   0.0226 *
I(rain^3)    3.932e-08  1.635e-08   2.405   0.0225 *

> (74.212 - 62.635)/3/1.94
[1] 1.989175
> pchisq(.Last.value, 3, lower.tail = F)
[1] 0.008982002
> (74.212 - 62.635)/3/1.94
[1] 1.989175
```

Estimating functions and quasi-likelihood

- suppose we assume only that
  $E(Y_j) = \mu_j(\beta), \quad \text{Var}(Y_j) = \phi_j V(\mu_j)$, as in glm
- and we use the glm estimates of β, defined by the score equation

$$g(Y;\beta) = \sum_{j=1}^{n} \frac{y_j - \mu_j}{\phi_j V(\mu_j)} \frac{x_{jr}}{g'(\mu_j)} = 0$$

- n.b. Davison calls LHS $g(Y;\beta)$, different g
- using only (*), we have

$$E\{g(Y;\beta)\} = 0; \qquad E\{-\frac{\partial g(Y;\beta)}{\partial \beta}\} = \text{Var}\{g(Y;\beta)\}$$

- thus g has two properties in common with the the score function from a log-likelihood

... estimating functions and quasi-likelihood

- $$g(Y;\beta) = \sum_{j=1}^{n} \frac{y_j - \mu_j}{\phi_j V(\mu_j)} \frac{x_{jr}}{g'(\mu_j)} = 0$$
- leads to $\tilde{\beta} \dot{\sim} N(\beta, (X^T W X)^{-1})$, where $W = \mathrm{diag}(w_i)$
- $w_i = 1/\{g'(\mu_j)^2 \phi_j V(\mu_j)\}$
- as usual, assume $\phi_j = \phi a_j$
- example: weighted LS – $V(\mu) = 1$, $weight_j = 1/a_j$
- example: constant coefficient of variation: $V(\mu) = \mu^2$
- example: overdispersed binomial or Poisson

... estimating functions and quasi-likelihood

- even if $V(\mu)$ is incorrectly specified, $\tilde{\beta}$ is still consistent
- $$\text{a.Var}(\tilde{\beta}) = (X^T W X)^{-1} \text{Var}\{g(Y;\beta)\}(X^T W X)^{-1}$$
- often is well approximated by $(X^T W X)^{-1}$ in any case
- when extended to dependent data, called generalized estimating equation method
- assume $E(Y) = \mu(\beta)$; $\quad \text{Var}(Y) = V(\mu)$
- $$X^T V^{-1}(\mu)(Y - \mu) = 0 \qquad (*)$$
- in the dependent data case, there is no quasi-likelihood function for which (*) is the score function
- reference: Liang & Zeger (1986, Biometrika);