Generalized Linear Models (Ch. 7)

Example (single sheet)

32 rolls of fabric

(y_i, x_i) $i = 1, \dots, 32$ x_i = length
y_i = # of faults

linear regression assumes constant variance

y_i is a count ≥ 0; 'relatively' small

A natural model $y_i \sim \text{Poisson}(\mu_i)$

$$\mu_i = \beta_0 + \beta_1 x_i$$

$$\text{or } \log \mu_i = \beta_0 + \beta_1 x_i \Rightarrow \mu_i = e^{\beta_0 + \beta_1 x_i} = e^{\beta_0} \times e^{\beta_1 x_i}$$

multiplicative relationship

Example h.o.p.

34 cities rainfall (mm) prop$^{=}$ of +ve samples (y_i)
number tested m_i

$$y_i = \text{prop}^{=} = \frac{r_i}{m_i} \begin{array}{l} \leftarrow \# \text{ +ve's} \\ \leftarrow \text{total } \# \end{array} \qquad \underline{r_i \sim \text{Bin}(m_i, p_i)}$$

start here

$E y_i = p_i$ prob $(Y_i = 1)$

e.g. $p_i = \beta_0 + \beta_1 x_i$ $\leftarrow$ not as useful

or $\log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 x_i$ logistic transform

$\log(p/(1-p)) =$ <u>logit</u>

$\Phi^{-1}(p_i) = \beta_0 + \beta_1 x_i$ <u>probit</u>

Φ normal cdf

General formulation of the above:

Assume $(\underline{x}_i, y_i)_{i=1}^n$ data ; $\mu_i = E y_i$

to depend on x_i ; Assume we have a fn $\ell(\cdot)$

$\ell(\mu_i) = \underline{x}_i^T \underline{\beta}$ <u>Link function</u> $\ell(\cdot)$ in VR

$= \eta_i$ linear <u>predictor</u> $= g(\cdot)$ in HO

Binomial, with logit $\ell(v) = \log\frac{v}{1-v}$

Poisson with log $\ell(v) = \log(v)$

Normal usually $\ell(v) = v$

Dist^{n} (or density) for y_i

$$f(y_i;\theta_i,\phi) = \exp\left[\frac{A_i\{y_i\theta_i - \gamma(\theta_i)\}}{\phi} + \tau\left(y_i, \frac{\phi}{A_i}\right)\right]$$

(7.1)

- A_i is known (weight)
- ϕ is same $\forall i$ ← known (or unknown)
- θ_i (parameters) are related via $\underline{x}_i^T\underline{\beta} = \eta_i$
- $\gamma(\cdot)$ is a f^{n} of known form

Ex: $N(\mu_i, \sigma^2)$

$$f(y_i) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y_i-\mu_i)^2}$$

$$= \exp\left[-\frac{1}{2\sigma^2}y_i^2 + \frac{\mu_i y_i}{\sigma^2} - \frac{1}{2\sigma^2}\mu_i^2 - \ln\sigma - \frac{1}{2}\ln(2\pi)\right]$$

$$= \exp\left[\frac{y_i\mu_i - \frac{1}{2}\mu_i^2}{\sigma^2} \underbrace{- \frac{\frac{1}{2}y_i^2}{\sigma^2} - \ln\sigma - \frac{1}{2}\ln(2\pi)}_{\tau(y_i,\,\sigma^2)}\right]$$

$A_i = 1 \quad \phi = \sigma^2$

$\theta_i = \mu_i \quad \gamma(\theta_i) = \gamma(\mu_i) = \frac{1}{2}\mu_i^2$

Ex. $Po(\mu_i)$

$$f(y_i) = \mu_i^{y_i} e^{-\mu_i} / y_i!$$

$$= \exp\left[y_i \log\mu_i - \mu_i - \log(y_i!)\right]$$

$$\phi = 1 \qquad A_i = 1 \qquad \theta_i = \log \mu_i \qquad \gamma(\theta_i) = \mu_i = e^{\theta_i}$$

Ex $Bin(m_i, p_i)$

$$f(y_i) = \binom{m_i}{y_i} p_i^{y_i} (1-p_i)^{m_i - y_i}$$

$$= \exp\left[y_i \log\left(\frac{p_i}{1-p_i}\right) + m_i \log(1-p_i) - \log \binom{m_i}{y_i} \right]$$

$$r_i = y_i / m_i$$

$$f(r_i) = \exp\left[m_i r_i \log \frac{p_i}{1-p_i} + m_i \log(1-p_i) - \log \binom{m_i}{m_i r_i} \right]$$

$$A_i = m_i \qquad \phi = 1$$

$$\theta_i = \log\left(\frac{p_i}{1-p_i}\right) \qquad p_i = \frac{e^{\theta_i}}{1+e^{\theta_i}}$$

$$\gamma(\theta_i) = -\log\left(\frac{1}{1+e^{\theta_i}}\right) = \log(1+e^{\theta_i})$$

Ex. Gamma density

$$f(y_i; \mu_i, \beta) = \frac{1}{\Gamma(\beta)} \left(\frac{\beta}{\mu_i}\right)^{\beta} y_i^{\beta - 1} e^{-y_i \beta / \mu_i}$$

$$y_i > 0$$

$$= \exp\left[-\beta\left(\frac{y_i}{\mu_i}\right) + \beta\log\left(\frac{\beta}{\mu_i}\right) + (\beta-1)\log y_i - \log\Gamma(\beta)\right]$$

$$= \exp\left[\beta\left(\frac{y_i}{-\mu_i} - \log\mu_i\right) + \beta\log\beta + (\beta-1)\log y_i - \log\Gamma(\beta)\right]$$

$$\phi = \frac{1}{\beta} \quad A_i = 1 \quad \theta_i = -\frac{1}{\mu_i} \quad \gamma(\theta_i) = \log(\theta_i)$$

θ_i is related to μ_i ; μ_i related to $x_i^T\beta$

$$\ell(\mu_i) = x_i^T\beta$$

$\theta_i = \ldots$ eventually a function of β

In normal $\theta_i = x_i^T\beta$

Poisson $\theta_i = \log\mu_i = x_i^T\beta$ for $\ell(v) = \log(v)$

Bin $\theta_i = \text{logit}(p_i) = x_i^T\beta$ for $\ell(v) = \log\left(\frac{v}{1-v}\right)$

Gamma $\theta_i = -\frac{1}{\mu_i}$ (1) $\frac{1}{\mu_i} = x_i^T\beta$ $\ell(v) = \frac{1}{v}$

(2) $\log(\mu_i) = x_i^T\beta$ $\ell(v) = \log(v)$

$\theta_i = \eta_i$ but in G (2) $\theta_i = -e^{-x_i^T\beta}$

Properties of generalized linear models.

$$f(y_i) = \exp\left[A_i\{y_i\theta_i - \gamma(\theta_i)\}/\phi + c(y_i, \phi/A_i) \right]$$

$$E y_i = \underline{\gamma'(\theta_i) = \mu_i} \qquad \mu_i \Leftrightarrow \theta_i$$

$$\int f(y)\,dy = 1 = \int e^{\frac{1}{\phi}A(y\theta - \gamma(\theta)) + c(y, \phi/A)}\,dy$$

$$\frac{\partial}{\partial\theta}\int f(y)\,dy = 0 = \int \frac{A}{\phi}(y - \gamma'(\theta))\, e^{\frac{1}{\phi}A(y\theta - \gamma(\theta)) + c(y, \frac{\phi}{A})}\,dy$$

$$\Rightarrow \int y e^{\cdots}\,dy = \gamma'(\theta)\cdot \int e^{\cdots}\,dy$$

$$\Rightarrow E y = \gamma'(\theta)$$

$$\mathrm{var}(y_i) = \frac{\phi}{A_i}\gamma''(\theta_i) \qquad \text{same trick \underline{twice}}$$

$$= \frac{\phi}{A_i}\left(\frac{\partial \mu_i}{\partial \theta_i}\right) \qquad \Leftarrow \gamma''(\theta_i) \text{ is a fn of } \mu_i$$

$$V(\mu_i) = \gamma''(\theta_i(\mu_i)) \quad \text{is usual notation}$$

$$\underline{\mathrm{var}(y_i) = \frac{\phi}{A_i}V(\mu_i)} \qquad \underline{E(y_i) = \mu_i}$$