… derived quantities, i.i.d. sample

observed likelihood	$L(\theta; y) \propto \prod_{i=1}^{n} f(y_i; \theta)$
log-likelihood	$\ell(\theta; y) = \sum_{i=1}^{n} \log f(y; \theta) + a(y)$
score	$U(\theta) = \partial \ell(\theta; y)/\partial\theta = O_p(\sqrt{n})$
maximum likelihood estimate	$\hat{\theta} = \hat{\theta}(y) = \arg\sup_\theta \ell(\theta; y)$
Fisher information	$j(\hat{\theta}) = -\partial^2 \ell(\hat{\theta}; y)/\partial\theta\partial\theta^T = O_p(n)$
expected information	$i(\theta) = \mathrm{E}_\theta U(\theta) U(\theta)^T = O(n)$

Bartlett identities

$$
\begin{aligned}
1 &= \int f(y;\theta)dy \text{ endpoints not specified}\\
0 &= \frac{\partial}{\partial\theta}\int f(y;\theta)dy\\
&= \int \frac{\partial}{\partial\theta} f(y;\theta)dy \text{ but can't involve } \theta\\
&= \int \frac{\partial}{\partial\theta}\ell(\theta;y)f(y;\theta)dy = \mathrm{E}_\theta\{U(\theta;Y)\}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{\partial}{\partial\theta}\int \frac{\partial}{\partial\theta}\ell(\theta;y)f(y;\theta)dy\\
&= \int [\frac{\partial^2}{\partial\theta\partial\theta^{\mathrm{T}}}\ell(\theta;y) + \{\frac{\partial}{\partial\theta}\ell(\theta;y)\}\{\frac{\partial}{\partial\theta}\ell(\theta;y)\}^{\mathrm{T}}]f(y;\theta)dy\\
\Rightarrow \mathrm{E}_\theta\{U(\theta)U^{\mathrm{T}}(\theta)\} &= \mathrm{E}_\theta\{-\frac{\partial^2}{\partial\theta\partial\theta^{\mathrm{T}}}\ell(\theta;y)\}\\
i(\theta) &= \mathrm{E}_\theta\{j(\theta)\}
\end{aligned}
$$

p-value functions of θ

-

$$
\begin{aligned}
r_u(\theta) &= U(\theta)j^{-1/2}(\hat\theta) \stackrel{.}{\sim} N(0,1) \\
r_e(\theta) &= (\hat\theta - \theta)j^{1/2}(\hat\theta), \\
r(\theta) &= \text{sign}(\hat\theta - \theta)[2\{\ell(\hat\theta) - \ell(\theta)\}]^{1/2}
\end{aligned}
$$

- approximate pivotal quantities

$$\Pr\{r_u(\theta) \le r_u^0(\theta)\} \doteq \Phi\{r_u^0(\theta)\}$$

under sampling from the model $f(y;\theta) = f(y_1, \dots, y_n; \theta)$

- p-value function (of θ, for fixed data)

$$p_u(\theta) = \Phi\{r_u^0(\theta)\}$$

- similarly $p_e(\theta) = \Phi\{r_e(\theta)\}, \quad p_r(\theta) = \Phi\{r(\theta)\}$ are also p-value functions for θ, based on limiting dist'ns

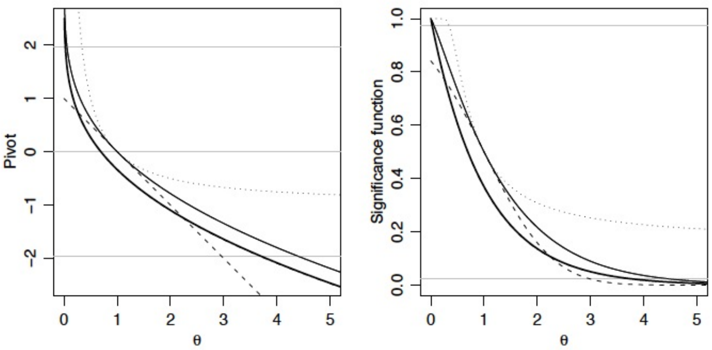

Figure 2.2: Approximate pivots and P-values based on an exponential sample of size $n = 1$. Left: likelihood root $r(\theta)$ (solid), score pivot $s(\theta)$ (dots), Wald pivot $t(\theta)$ (dashes), modified likelihood root $r^*(\theta)$ (heavy), and exact pivot $\theta \sum y_j$ (dot-dash). The modified likelihood root is indistinguishable from the exact pivot. The horizontal lines are at $0, \pm 1.96$. Right: corresponding significance functions, with horizontal lines at 0.025 and 0.975.

Nuisance parameters

- $\theta = (\psi, \lambda) = (\psi_1, \ldots, \psi_q, \lambda_1, \ldots, \lambda_{d-q})$
- $U(\theta) = \begin{pmatrix} U_\psi(\theta) \\ U_\lambda(\theta) \end{pmatrix}, \qquad U_\lambda(\psi, \hat{\lambda}_\psi) = 0$
- $i(\theta) = \begin{pmatrix} i_{\psi\psi} & i_{\psi\lambda} \\ i_{\lambda\psi} & i_{\lambda\lambda} \end{pmatrix} \quad j(\theta) = \begin{pmatrix} j_{\psi\psi} & j_{\psi\lambda} \\ j_{\lambda\psi} & j_{\lambda\lambda} \end{pmatrix}$
- $i^{-1}(\theta) = \begin{pmatrix} i^{\psi\psi} & i^{\psi\lambda} \\ i^{\lambda\psi} & i^{\lambda\lambda} \end{pmatrix} \quad j^{-1}(\theta) = \begin{pmatrix} j^{\psi\psi} & j^{\psi\lambda} \\ j^{\lambda\psi} & j^{\lambda\lambda} \end{pmatrix}.$
- $i^{\psi\psi}(\theta) = \{i_{\psi\psi}(\theta) - i_{\psi\lambda}(\theta) i_{\lambda\lambda}^{-1}(\theta) i_{\lambda\psi}(\theta)\}^{-1},$
- $\ell_{\mathrm{P}}(\psi) = \ell(\psi, \hat{\lambda}_\psi), \qquad j_{\mathrm{P}}(\psi) = -\ell''_{\mathrm{P}}(\psi)$

Inference from limiting distributions, nuisance parameters

$$\begin{aligned} w_u(\psi) = U_\psi(\psi,\hat\lambda_\psi)^T\{i^{\psi\psi}(\psi,\hat\lambda_\psi)\}U_\psi(\psi,\hat\lambda_\psi) &\quad\dot\sim\quad \chi^2_q \\ w_e(\psi) = (\hat\psi-\psi)\{i^{\psi\psi}(\hat\psi,\hat\lambda)\}^{-1}(\hat\psi-\psi) &\quad\dot\sim\quad \chi^2_q \\ w(\psi) = 2\{\ell(\hat\psi,\hat\lambda)-\ell(\psi,\hat\lambda_\psi)\} = 2\{\ell_{\mathrm{P}}(\hat\psi)-\ell_{\mathrm{P}}(\psi)\} &\quad\dot\sim\quad \chi^2_q; \end{aligned}$$

Approximate Pivots, $q = 1$

$$\begin{aligned} r_u(\psi) &= \ell_{\mathrm{P}}'(\psi)j_{\mathrm{P}}(\hat\psi)^{1/2}\ \dot\sim\ N(0,1), \\ r_e(\psi) &= (\hat\psi-\psi)j_{\mathrm{P}}(\hat\psi)^{1/2}\ \dot\sim\ N(0,1), \\ r(\psi) &= \operatorname{sign}(\hat\psi-\psi)2\{\ell_{\mathrm{P}}(\hat\psi)-\ell_{\mathrm{P}}(\psi)\}\ \dot\sim\ N(0,1) \end{aligned}$$

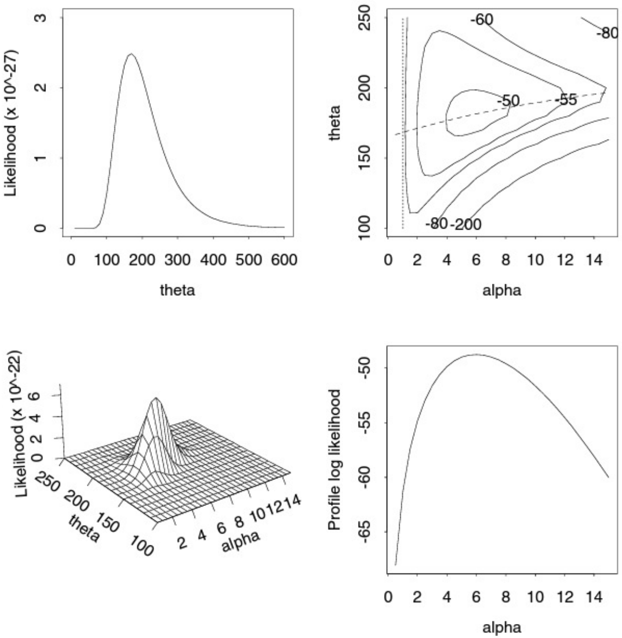

Figure 4.1 Likelihoods for the spring failure data at stress 950 N/mm^2. The upper left panel is the likelihood for the exponential model, and below it is a perspective plot of the likelihood for the Weibull model. The upper right panel shows contours of the log likelihood for the Weibull model; the exponential likelihood is obtained by setting $\alpha = 1$. that is, slicing L along the vertical dotted line. The lower right panel shows the profile log likelihood for α, which corresponds to the log likelihood values along the dashed line in the panel above, plotted against α.

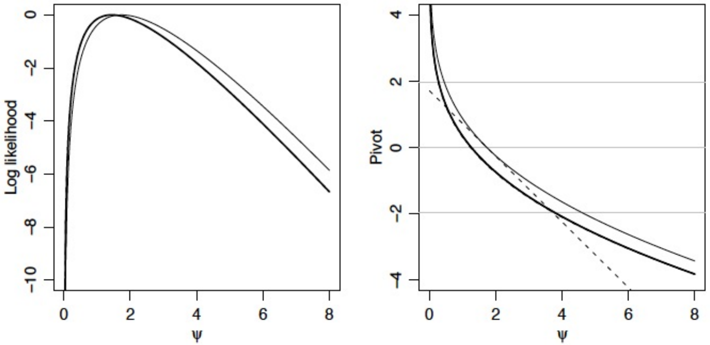

Figure 2.3: Inference for shape parameter ψ of gamma sample of size $n = 5$. Left: profile log likelihood $\ell_{\rm p}$ (solid) and the log likelihood from the conditional density of u given v (heavy). Right: likelihood root $r(\psi)$ (solid), Wald pivot $t(\psi)$ (dashes), modified likelihood root $r^*(\psi)$ (heavy), and exact pivot overlying $r^*(\psi)$. The horizontal lines are at $0, \pm 1.96$.