Exercises October 30 **STA 4508S (Fall, 2018)**

1. Suppose that Y_i are independent exponential random variables with $\mathrm{E}(Y_i) = \psi\lambda_i$, and Z_i are independent exponential random variables with $\mathrm{E}(Z_i) = \psi/\lambda_i, \quad i = 1, \ldots, n$.

   (a) Show that ψ is orthogonal to $\lambda = (\lambda_1, \ldots, \lambda_n)$, with respect to expected Fisher information.

   (b) Find the maximum likelihood estimates of λ and ψ.

   (c) Show that $\hat{\psi}$ is not consistent for ψ as $n \to \infty$.

   (d) Derive the adjusted profile log-likelihood for ψ:

   $$\ell_{\mathrm{a}}(\psi) = \ell_{\mathrm{p}}(\psi) - \frac{1}{2}\log|j_{\lambda\lambda}(\psi, \hat{\lambda}_\psi)|,$$

   and find the probability limit of the adjusted maximum likelihood estimator $\hat{\psi}_{\mathrm{a}}$ obtained by maximizing $\ell_{\mathrm{a}}(\psi)$.

2. (Knight, 2000 Ch. 5.6; Owen, 1988). Suppose $Y_1, \ldots, Y_n$ are independent and identically distributed from an unknown distribution function F. To estimate F we restrict attention to distributions putting positive probability mass only at the points $Y_1, \ldots, Y_n$, assumed distinct. Knight defines the non-parametric log-likelihood function for $F(\cdot)$ as

   $$L(p_1, \ldots, p_n) = \sum_{i=1}^{n} \log(p_i), \quad p_i \geq 0, \Sigma p_i = 1,$$

   where p_i is the probability mass at Y_i.

   (a) Show that $L(p)$ (or equivalently $\ell(p) = \log L(p)$) is maximized at $\hat{p}_i = 1/n$.

   (b) Suppose that $\mu = E(Y_i) = \int y dF(y)$ is the parameter of interest, with $F(\cdot)$ as a nuisance parameter. The profile likelihood is obtained by maximizing

   $$L(p_1, \ldots, p_n), \text{ subject to } p_i \geq 0, \Sigma p_i = 1, \Sigma p_i Y_i = \mu,$$

where there is now an additional constraint on the vector p. Show that the solution to the maximization problem is given by

$$\begin{aligned}\hat{p}_i(\mu) &= \frac{1}{n}\frac{1}{1+\lambda(Y_i-\mu)}, \text{ where } \lambda \text{ solves} \\ 0 &= \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i-\mu}{1+\lambda(Y_i-\mu)}.\end{aligned}$$