Accurate semi-Lagrangian time stepping for gas storage problems

Tony Ware
University of Calgary

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Natural gas storage and its role in the market

Types of storage

- **depleted fields** - plentiful, high base gas, slow cycling
- **aquifers** - very high base gas, close to consumption, slow cycling
- **salt domes** - limited in size, low base gas, fast cycling
- **pipelines** - varying characteristics, balancing and peak services
- **LNG** - peak shaving services

Source: industrialgasplants.com
Natural gas storage and its role in the market

- Storage acts as a buffer between wellhead gas supply and pipeline flows and demand, which is highly variable.
- Storage turnover accounts for a significant proportion of total consumption.
- Demand is typically higher in winter and lower in summer, so gas is typically injected into storage in the summer and withdrawn in winter.

Source: EIA

![Graph showing US Natural Gas Consumption by End Use (Monthly)](http://tonto.eia.doe.gov/dnav/ng/ng_cons_sum_dcu_nus_m.htm)

Source: EIA
Natural gas storage
Storage in the market

Working gas in underground storage in US (lower 48)

- Consuming region west
- Producing region
- Consuming region east
- Total

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Natural gas storage and its role in the market
Natural gas storage

The management of storage

The optimal management of storage depends on the constraints (physical, network, regulatory) that apply to the facility, and on the goals of the operator.

- LDCs are required to maintain storage as a guarantee of supply. In recent years they’ve been able to unbundle and delegate operation of facilities to third parties, who can manage the facility with the goal of making profit in the gas market.

- Storage facilities may be associated with particular end uses - power generation for instance - and the facility will need to be managed accordingly. Market-wide, the growth of gas-fired generation has increased the need for storage with high deliverability.

- Storage owners may lease the facility to a range of third parties in the form of park&loan agreements of various kinds. In this case they manage the facility as a hedge for their contractual commitments.
Storage valuation and optimal policy determination

Intrinsic and extrinsic value

The *intrinsic value* of storage arises from seasonality, and is the maximum value that can be locked in by using the storage as a hedge for a shaped forward position. The *extrinsic value* comes from the flexibility to respond to movements in prices as they unfold.

Heuristic approaches

Rolling intrinsic valuation  Maragos (2002); Gray and Khandelwal (2004); Bjerksund and Vagstad (2008); Lai et al. (2010); Wu et al. (2010)

Calendar spreads  Eydeland and Wolyniec (2003); Gray and Khandelwal (2004)
Storage valuation and optimal policy determination

Stochastic dynamic programming - spot trading

- The dynamic programming principle can be used to characterize the optimal storage value as a function of the current price(s) in the market.
- The control variable is the injection/withdrawal rate, determined according to some strategy.
- The optimal strategy is the one that maximizes the discounted risk-neutral expected value of the resulting cashflows over the lifetime of the contract.
- Gas is traded on the spot market, but the use of a risk-neutral expectation assumes the existence of a liquid forward market that can be used for hedging.
- Physical characteristics of various kinds may be incorporated.
We let $\theta_t$ denote the injection(-)/withdrawal(+) rate (at the wellhead). This is related to the storage level $q_t$ via

$$dq_t = \psi(q_t, \theta_t)dt.$$ 

At any given time, there is a set of feasible values of the control $\theta_t$:

$$\theta_t \in \Theta(t, q_t).$$ 

The global constraints on $q_t$ we assume to be of the form

$$q_t \in Q,$$

where $Q = [A, B]$ is a (possibly time-dependent) interval in $[0, \infty)$.

There is an instantaneous rate of cash flow associated with injection/withdrawal. This will be combined with a rental rate, and each of these may depend on the current markets gas price(s) $x_t$, and on the level of storage. Thus the net instantaneous rate of cash flow (inward) may be denoted by $\phi(x_t, q_t, \theta_t)$. 


We assume that an injection/withdrawal strategy will take the form of a function \( \theta(t, x, q) \) such that \( \theta_t = \theta(t, x_t, q_t) \).

Given such a policy, we can define the discounted expected cashflow:

\[
V_\theta(t, x, q) = \mathbb{E} \left[ \int_t^T \phi(x_s, q_s, \theta_s) \, ds + h(x_T, q_T) \mid x_t = x, q_t = q \right],
\]

where \( h \) is a terminal value function (zero interest rates, for simplicity).

The optimal value is then

\[
V(t, x, q) := \sup_{\theta \in \Theta} V_\theta(t, x, q).
\]

\( V \) defined in this way satisfies the HJB equation

\[
\frac{\partial V}{\partial t} + \sup_{\theta \in \Theta(t, q)} \left\{ \mathcal{L}(t, x) V + \psi(q, \theta) \frac{\partial V}{\partial q} + \phi(x, q, \theta) \right\} = 0,
\]

where \( \mathcal{L}(t, x) \) is the infinitesimal generator of the process followed by \( x_t \).
Bellman’s principle of optimality leads to approaches based on
- trinomial trees [Ahn et al. (2002); Parsons (2005); Manoliu (2004)]
- Monte Carlo methods [Barrera-Esteve et al. (2006); Boogert and de Jong (2007); Holland (2007); Carmona and Ludkovski (2010)].

The HJB equation has been typically been solved using finite difference methods [Weston (2002); Thompson et al. (2009); Chen and Forsyth (2007)].
- Thompson et al. (2009) used a TVD scheme to deal with the $\frac{\partial V}{\partial q}$ term: without this, spurious oscillations polluted the solution.
- Chen and Forsyth (2007) used a semi-Lagrangian approach to deal with this term.
The semi-Lagrangian method combines terms such as $\frac{\partial V}{\partial t} + \psi(q)\frac{\partial V}{\partial q}$ into a single directional derivative $D_t V$, defined by

$$D_t V = \frac{d}{ds} V(s, x, Q(s)) \bigg|_{s=t},$$

where $Q(s)$ satisfies the ODE

$$\frac{d}{ds} Q(s) = \psi(Q(s)); \quad Q(t) = q.$$

In our case the function $\psi$ depends also on the policy $\theta$, and we could write the HJB equation as

$$\sup_{\theta \in \Theta(t, q)} \left\{ D^\theta_t V + \mathcal{L}(t, x)V + \phi(x, q, \theta) \right\} = 0.$$

It turns out to be more fruitful to delay the optimization step and, given a policy $\theta$, start with the equation satisfied by $V^\theta$:

$$\frac{\partial V^\theta}{\partial t} + \mathcal{L}(t, x)V^\theta + \psi(q, \theta(t, x, q)) \frac{\partial V^\theta}{\partial q} + \phi(x, q, \theta(t, x, q)) = 0.$$
For each $t, x, q$, let $Q_\theta(\cdot)$ be defined by

$$\frac{d}{ds} Q_\theta(s) = \psi \left( Q_\theta(s), \theta(s, x, Q_\theta(s)) \right); \quad Q_\theta(t) = q.$$ 

Then, in Lagrangian form,

$$\frac{d}{ds} V_\theta(s, x, Q_\theta(s)) + \mathcal{L}(s, x)V_\theta(s, x, Q_\theta(s)) + \phi(x, Q_\theta(s), \theta(s)) = 0.$$ 

If we integrate between $t^n$ and $t^{n+1}$, and take $t = t^n$ in the definition of $Q_\theta$, we obtain

$$V_\theta(t^n, x, q) - V_\theta(t^{n+1}, x, Q_\theta(t^{n+1}))$$

$$= \int_{t^n}^{t^{n+1}} \left[ \mathcal{L}V_\theta(s, x, Q_\theta(s)) + \phi(x, Q_\theta(s), \theta(s, x, Q_\theta(s))) \right] ds,$$

where, for simplicity, we have dropped the explicit dependence of $\mathcal{L}$ on $t$ and $x$. 

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Semi-Lagrangian timestepping

- We approximate, using the trapezium rule,

\[ \int_{t_n}^{t_{n+1}} \mathcal{L} V_\theta(s, s, Q_\theta(s)) \, ds \approx \frac{\Delta t}{2} \left( \mathcal{L} V^n_\theta(x, q) + \mathcal{L} V^{n+1}_\theta(x, Q_\theta(t^{n+1})) \right) , \]

leaving us with

\[ (I - \frac{\Delta t}{2} \mathcal{L}) V_\theta(t^n, x, q) \approx (I + \frac{\Delta t}{2} \mathcal{L}) V_\theta(t^{n+1}, x, Q_\theta(t^{n+1})) \]

\[ + \int_{t_n}^{t_{n+1}} \phi \left( x, Q_\theta(s), \theta(s, x, Q_\theta(s)) \right) \, ds \]

- Guided by this approximation we define the sequence of functions \( V^n_\theta(x, q) \) to satisfy \( V^N_\theta(x, q) = h(x, q) \) and, for \( n < N = T/\Delta t \),

\[ (I - \frac{\Delta t}{2} \mathcal{L}) V^n_\theta(x, q) = \sup_\theta \left\{ (I + \frac{\Delta t}{2} \mathcal{L}) V^{n+1}_\theta(x, Q_\theta(t^{n+1})) \right\} \]

\[ + \int_{t_n}^{t_{n+1}} \phi \left( x, Q_\theta(s), \theta(s, x, Q_\theta(s)) \right) \, ds \right\}. \]
The policies $\theta$ we are optimizing are restricted to attainable policies in the time interval $[t^n, t^{n+1}]$. We factor this set into the set $Q^{n+1}_n$ of attainable values $Q$ of $Q_\theta(t^{n+1})$, and the set $\Theta^Q_q$ of policies that lead from $q$ to $Q$.

- For each $Q \in Q^{n+1}_n$, we find the most profitable way to attain it:

$$
\chi^{n+1}_n(Q; x, q) := \max_\theta \int_{t^n}^{t^{n+1}} \phi(x, Q_\theta(s), \theta(s, x, Q_\theta(s))) \, ds.
$$

- Then we can optimise over $Q$ to define our optimal solution $V^n$ from the ‘previous’ value $V^{n+1}$:

$$
(I - \frac{\Delta t}{2} \mathcal{L}) V^n(x, q) = \sup_{Q \in Q^{n+1}_n} \left\{ (I + \frac{\Delta t}{2} \mathcal{L}) V^{n+1}(x, Q) + \chi^{n+1}_n(Q; x, q) \right\}.
$$

- The framework of Barles and Souganidis (1991) can be used to show that, under certain conditions on $\mathcal{L}$ and $\phi$, the solution to this semi-discrete equation converges to the unique viscosity solution of the HJB equation (sketch proof).
A specific storage setting

We consider a setting introduced by Thompson, Davison and Rasmussen (2009), and used by Chen and Forsyth (2009). Here we have

\[ \psi(q, \theta) = \begin{cases} \theta & \text{if } \theta \leq 0 \quad \text{(withdrawal)} \\ \theta - k & \text{if } \theta > 0 \quad \text{(injection)} \end{cases} \]

We also have

\[ \phi(x, q, \theta) = \begin{cases} -x\theta & \text{if } \theta \leq 0 \\ -x(\theta + k) & \text{if } \theta > 0 \end{cases} \]

Importantly, they give engineering-based \( q \)-dependent constraints on the allowable rates of injection/withdrawal:

\[ \theta \in [-\bar{\theta}_-, \bar{\theta}_+] = \left[ -k_1 \sqrt{q}, k_2 \sqrt{\frac{1}{q + k_3} - \frac{1}{k_4}} \right]. \]
A specific storage setting

Optimal exercise between timesteps

In this setting, we have

\[
\chi^{n+1}_n(Q; x, q) = -x \min_\theta \int_{t^n}^{t^{n+1}} \theta(s) + k1_{\{\theta(s) > 0\}} ds.
\]

The minimum is over all strategies that get us from \(q\) to \(Q\), which must therefore satisfy

\[
\int_{t^n}^{t^{n+1}} \psi(Q\theta(s), \theta(s)) ds = \int_{t^n}^{t^{n+1}} \theta(s) - k1_{\{\theta(s) > 0\}} ds = Q - q.
\]

If we assume that \(\bar{\theta}_\pm\) are approximately constant over the course of a single timestep, we can solve this maximisation problem explicitly:

\[
\chi^{n+1}_n(Q; x, q) = -x(Q - q) - \frac{2xk}{\theta_+ - k}(Q - q)_+.
\]

Without this assumption, we need to perform a separate computation to find the function \(\chi^{n+1}_n\).
Once we have determined the function $\chi_{n+1}^n(Q; x, q)$, as well as $Q_{n+1}^n$, the maximum attainable range of values for $Q$ given a starting level of $q$, we can solve the recursive optimisation to find our optimal solution $V^n$ from the ‘previous’ value $V^{n+1}$:

\[
(I - \frac{\Delta t}{2} \mathcal{L}) V^n(x, q) = \sup_{Q \in Q_{n+1}^n} \left\{ (I + \frac{\Delta t}{2} \mathcal{L}) V^{n+1}(x, Q) + \chi_{n+1}^n(Q; x, q) \right\}.
\]

Notice that the semi-Lagrangian discretisation has decoupled the search for the optimal solution into two problems.

1. Find the most profitable way to get from a storage level of $q$ to one of $Q$ over the course of a time step.
2. For each $q$, find the value of $Q$ that maximises the storage value.
Full discretization

- \( q \) dimension: lay down \( N_Q + 1 \) points on a grid from \( A \) to \( B \), and linearly interpolate between values at these points. In the storage setting considered here, these points are *stretched* so that they congregate near the minimum and maximum storage levels.

- \( x \) dimension: use Fourier cosine discretization with \( N_X \) points.
  - With the use of the FFT this enables constant-coefficient intergro-differential operators to be implemented in \( O(N \log N) \) operations.
  - The domain is truncated to \([x_L, x_R]\), and the half-periodicity effectively means that the solution is repeatedly reflected about the endpoints.
  - The method is an adaptation of the Fourier space timestepping approach championed by Jackson et al. (2008), and the CONV method of Fang and Oosterlee (2008).

- Non-constant drift coefficients are dealt with by using a semi-Lagrangian formulation in \( x \) as well as in \( q \) (after taking a log-transform if necessary).
Two price models

To give an example of how the method performs, we’ll use two models based on those used by Chen and Forsyth (2007).

1. Mean-reversion with diffusion:

\[ dx_t = 2.38(6 - x_t)dt + 0.59x_t dW(t). \]

2. Mean-reversion with jump-diffusion:

\[ dx_t = 2.38(6 - x_t)dt + 0.59x_t dW(t) + (\eta - 1)x_t dJ(t), \]

where \( dJ(t) \) is an independent Poisson process with intensity \( \lambda = 12 \), and \( \eta \) is lognormal with density

\[ g(\eta) = \frac{1}{\sqrt{2\pi}\gamma\eta} \exp \left( -\frac{(\ln \eta - \nu)^2}{2\gamma^2} \right), \]

with \( \gamma = 0.198 \) and \( \nu = -0.0196 \), chosen to satisfy \( \mathbb{E}[\eta - 1] = 0 \).
A specific storage setting

Parameters

- The storage valuation runs over three years. The riskless discount rate in the simulation is 10%.
- The storage level can range from 0 to 2000 MMcf. The remaining parameters are:

  \[ k_1 = 2040.41, \ k_2 = 730000, \ k_3 = 500, \ k_4 = 2500, \ k = 1.7 \times 365. \]

- A penalty is imposed at the end of the contract if there is not enough gas left in storage:

  \[ h(x, q) = -2000x(1000 - q)_+. \]

- The truncated range of asset prices \( S = \ln x \) is from 0.1 to 360 $/MMBtu.
- The values reported in the convergence results correspond to a storage level of 1000 MMcf and $6/MMBtu.
Typical value surface
Control surface: $N_T = 320$, $N_X = 256$, $N_Q = 32$

The controls are not bang-bang
Control surface: $N_T = 1280, N_X = 1280, N_Q = 128$

But they converge towards bang-bang controls as the grid is refined.

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Example storage problem
Convergence

$N_T = 10N$, $N_X = 8N$, $N_Q = N$. Reference line: second order
Convergence (with jumps)

\[ N_T = 10N, \, N_X = 8N, \, N_Q = N. \] Reference line: second order

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Example storage problem
Convergence: bang-bang controls

$N_T = 10N$, $N_X = 8N$, $N_Q = N$. Reference line: second order
Concluding Remarks

- The semi-Lagrangian approach is fundamentally an approach to time discretization; as such it can be combined with any desired discretization in the other dimensions, including Monte Carlo.

- Second order accuracy can be achieved for certain kinds of stochastic optimal control problems.

- Although I did not discuss seasonality, it can be incorporated without altering the format of the equations, although care should be taken to limit the amount of time spent determining the functions $\chi_{n+1}^n(Q; x, q)$.

- Multifactor models are essential for any realistic storage valuation: unless the market model is rich enough to capture the variability of different calendar spreads, much of the value will be lost.

- Other challenges involve addressing some of the considerations faced by actual storage managers: such as determining the optimal policy when the storage facility is used as a partial hedge for existing risks, for example when subleasing the storage facility.
References I


