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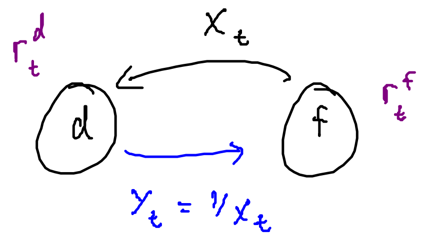
(CAD / USD)

FX options :
(exchange)



$$\frac{dX_t}{X_t} = \mu_t \, dt + \sigma_t \, dW_t$$

W_t: $\mathbb{P}$ - B.mtn

μ_t could be e.g. $\kappa(\theta - \ln X_t)$

domestic MM : $\dfrac{dM_t^d}{M_t^d} = r_t^d \, dt$

Foreign MM : $\dfrac{dM_t^f}{M_t^f} = r_t^f \, dt$

$M_t^f X_t$ is how much the Foreign account is worth to domestic investor.

$\mathbb{Q}^d$ the domestic risk-neutral measure

$$d(M_t^f X_t) = dM_t^f \, X_t + M_t^f \, dX_t + \underbrace{d[M^f, X]_t}_{\to 0}$$

$$= r_t^f \, M_t X_t \, dt + M_t^f X_t (\mu_t \, dt + \sigma_t \, dW_t)$$

$$\Rightarrow \quad \frac{d(M_t^f X_t)}{(M_t^f X_t)} = (\mu_t + r_t^f) \, dt + \sigma_t \, dW_t$$

$$= r_t^d \, dt + \sigma_t \, dW_t^d$$

b/c all domestic traded assets grow at r_t^d under the $\mathbb{Q}^d$ - measure.

Girsanov's Theorem then says:

$$W_t^d = \int_0^t \underbrace{\frac{(\mu_u + r_u^f - r_u^d)}{\sigma_u}}_{\lambda_u^x} du + W_t$$

is a $\mathbb{Q}^d$ - mtg where:

$$\left(\frac{d\mathbb{Q}^d}{d\mathbb{P}}\right)_T = \exp\left\{ -\frac{1}{2}\int_0^T (\lambda_u^x)^2\,du - \int_0^T \lambda_u^x\,dW_u \right\}.$$

So then:

$$\frac{dX_t}{X_t} = \mu_t\,dt + \sigma_t\,dW_t$$

$$= \mu_t\,dt + \sigma_t\left(-\frac{(\mu_t + r_t^f - r_t^d)}{\sigma_t}\,dt + dW_t^d \right)$$

$$\Rightarrow \boxed{\frac{dX_t}{X_t} = (r_t^d - r_t^f)\,dt + \sigma_t\,dW_t^d}$$

$f(t,x) = 1/x$

$$dY_t = d(1/X_t)$$ using $\mathbb{P}$-Brownian $\frac{dX_t}{X_t} = \mu_t\,dt + \sigma_t\,dW_t$

$$= d\,f(t, X_t)$$

$$= (\partial_t f + \mu_t X_t\,\partial_x f + \tfrac{1}{2}\sigma_t^2 X_t^2\,\partial_{xx} f)\,dt$$

$$+ \sigma_t X_t\,\partial_x f\,dW_t$$

$$= \left[0 + \mu_t X_t\left(-\frac{1}{X_t^2}\right) + \tfrac{1}{2}\sigma_t^2 X_t^2\left(+\frac{2}{X_t^3}\right)\right] dt$$

$$+ \sigma_t X_t\left(-\frac{1}{X_t^2}\right) dW_t$$

$$\Rightarrow \frac{dY_t}{Y_t} = (-\mu_t + \sigma_t^2)\,dt - \sigma_t\,dW_t$$

$$= (r_t^f - r_t^d)\,dt - \sigma_t\,\underbrace{dW_t^f}_{\mathbb{Q}^f}$$

How about dY_t in terms of $\mathbb{Q}^d$ - B.motn?

i) $$\frac{dY_t}{Y_t} = (-\mu_t + \sigma_t^2)\,dt - \sigma_t\,dW_t$$

$$= (-\mu_t + \sigma_t^2)\,dt - \sigma_t\left(-\frac{(\mu_t - r_t^d + r_t^F)}{\sigma_t}\,dt + dW_t^d\right)$$

$$\boxed{\frac{dY_t}{Y_t} = (r_t^F - r_t^d + \sigma_t^2)\,dt - \sigma_t\,dW_t^d}$$

$$\Rightarrow \quad W_t^F = -\int_0^t \sigma_u\,du + W_t^d \quad \longleftarrow ?$$

$$\left(\frac{d\mathbb{Q}^F}{d\mathbb{Q}^d}\right)_T = \exp\left\{-\frac{1}{2}\int_0^T \sigma_u^2\,du + \int_0^T \sigma_u\,dW_u^d\right\}$$

$$dW_t^d = \cancel{-}\,\frac{\mu_t - r_t^d + r_t^F}{\sigma_t}\,dt + dW_t$$

$$\begin{tikzcd} & \mathbb{P} \arrow[dl] \arrow[dr] & \\ \mathbb{Q}^d \arrow[ur] \arrow[rr] & & \mathbb{Q}^F \end{tikzcd}$$

$$dW_t^F = \cancel{-}\,\frac{-\mu_t + \sigma_t^2 - r_t^F + r_t^d}{-\sigma_t}\,dt + dW_t$$

$$dW_t^F = -\left(\frac{-\mu_t + \sigma_t^2 - r_t^F + r_t^d}{\sigma_t} + \frac{\mu_t - r_t^d + r_t^F}{\sigma_t}\right)dt + dW_t^d$$

$$= -\sigma_t\,dt + dW_t^d$$

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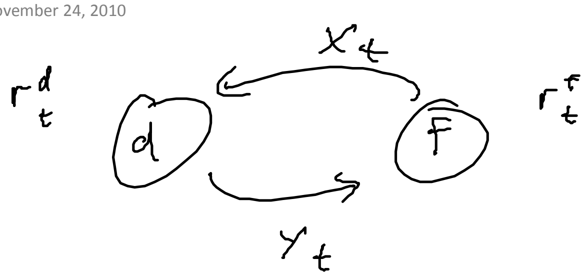

$$\frac{dX_t}{X_t} = \mu_t \, dt + \sigma_t \, dW_t$$

$\hookrightarrow$ $\mathbb{P}$ - B. motion

$$= (r_t^d - r_t^f)\, dt + \sigma_t \, dW_t^d$$

$\llcorner$ $\mathbb{Q}^d$ - B. motion.

consider an option paying $(X_T - \alpha)_+ \, F$

$\llcorner$ Foreign \$

so $$\frac{V_t}{M_t^d} = \mathbb{E}_t^{\mathbb{Q}^d}\left[\frac{(X_T - \alpha)_+ \, F}{M_T^d} \right]$$

$$\Rightarrow V_t = e^{-r^d (T-t)} \, \mathbb{E}_t^{\mathbb{Q}^d}\left[(X_T - \alpha)_+ F \right]$$

and $$\frac{dX_t}{X_t} = (r^d - r^f)\, dt + \sigma \, dW_t^d$$

$$\Rightarrow X_T \overset{d}{=} X_t \, e^{(r^d - r^f - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{(T-t)}\, Z}$$

$$Z \underset{\mathbb{Q}^d}{\sim} N(0,1)$$

$$\Rightarrow V_t = e^{-r^f (T-t)} \left(e^{-(r^d - r^f)(T-t)} \, \mathbb{E}_t^{\mathbb{Q}^d}\left[(X_T - \alpha)_+ F \right] \right)$$

$$= F e^{-r^f (T-t)} \left(X_t \, \Phi(d_+) - \alpha \, e^{-(r^d - r^f)(T-t)} \, \Phi(d_-) \right)$$

$$d_\pm = \frac{\ln(X_t/\alpha) + (r^d - r^f \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

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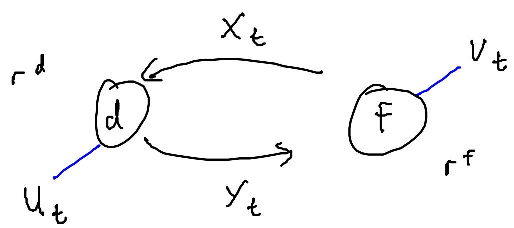

$\mathbb{P}$ - B.mtn

assume: $\dfrac{dV_t}{V_t} = \mu\, dt + \sigma\, dB_t$

$d[B,W]_t = \rho\, dt$

$\dfrac{dX_t}{X_t} = \alpha\, dt + \eta\, dW_t$

$X_t V_t$ is the domestic price of the foreign asset and is therefore a "domestic asset"

$\Rightarrow \quad d(X_t V_t) = dX_t\, V_t + X_t\, dV_t + d[X,V]_t$

$\Rightarrow \quad \dfrac{d(X_t V_t)}{(X_t V_t)} = (\alpha\, dt + \eta\, dW_t) + (\mu\, dt + \sigma\, dB_t)$

$+ \sigma\eta\rho\, dt$

$= (\alpha + \mu + \sigma\eta\rho)\, dt + \eta\, dW_t + \sigma\, dB_t$

$= r^d\, dt + \eta\, dW_t^d + \sigma\, dB_t^d$

write $B_t = \rho\, W_t + \sqrt{1-\rho^2}\; W_t^{\perp}$

$B_t^d = \rho\, W_t^d + \sqrt{1-\rho^2}\; W_t^{d\perp}$

recall that $dW_t^d = \dfrac{\alpha - r^d + r^f}{\eta}\, dt + dW_t$

write $dW_t^{d\perp} = \gamma\, dt + dW_t^{\perp}$

goal: Find γ.

$\cancel{\dfrac{dV_t}{V_t}} = \dfrac{d(X_t V_t)}{X_t V_t}$

$(\alpha + \mu + \sigma\eta\rho)\, dt + \eta\, dW_t + \sigma\, dB_t$

$= (\alpha + \mu + \sigma\eta\rho)\, dt + \eta\, dW_t + \sigma\left(\rho\, dW_t + \sqrt{1-\rho^2}\; dW_t^{\perp}\right)$

$$= (\alpha + \mu + \sigma\eta\rho)\, dt + (\eta + \sigma\rho)\, dW_t + \sigma\sqrt{1-\rho^2}\, dW_t^{\perp}$$

$$= (\alpha + \mu + \sigma\eta\rho)\, dt + (\eta + \sigma\rho)\left(dW_t^d - \left(\frac{\alpha - r^d + r^f}{\eta}\right) dt\right)$$

$$+ \sigma\sqrt{1-\rho^2}\left(dW_t^{d\perp} - \gamma\, dt\right)$$

$$= \underbrace{\left(r^d - r^f + \mu + \sigma\rho\left[\eta - \frac{\alpha - r^d + r^f}{\eta}\right] - \gamma\,\sigma\sqrt{1-\rho^2}\right)}_{= r^d} dt$$

$$+ \underbrace{(\eta + \sigma\rho)\, dW_t^d + \sigma\sqrt{1-\rho^2}\, dW_t^{d\perp}}_{\eta\, dW_t^d + \sigma\, dB_t^d}$$

$$\therefore \quad \frac{dV_t}{V_t} = \mu\, dt + \sigma\, dB_t$$

$$= \mu\, dt + \sigma\left(\rho\, dW_t + \sqrt{1-\rho^2}\, dW_t^{\perp}\right)$$

$$= \mu\, dt + \sigma\left[\rho\left(dW_t^d - \frac{\alpha - r^d + r^f}{\eta}\, dt\right) + \sqrt{1-\rho^2}\left(dW_t^{d\perp} - \gamma\, dt\right)\right]$$

$$= \underbrace{\left[\mu - \sigma\rho\, \frac{\alpha - r^d + r^f}{\eta} - \sigma\sqrt{1-\rho^2}\,\gamma\right]}_{= r^f - \sigma\rho\eta} dt + \sigma\, dB_t^d$$

b/c

$$\Rightarrow \boxed{\frac{dV_t}{V_t} = (r^f - \sigma\rho\eta)\, dt + \sigma\, dB_t^d}$$

$$\frac{dV_t}{V_t} = r^f\, dt + \sigma\, dB_t^f$$

recall $dW_t^f = -\eta\, dt + dW_t^d$

between $\mathbb{Q}^d \longleftrightarrow \mathbb{Q}^f$ only W_t changes not $W_t^{\perp}$

$$dB_t^F = \rho \, dW_t^F + \sqrt{1-\rho^2} \, dW_t^{F\perp}$$

$$= -\rho\eta \, dt + \rho \, dW_t^d + \sqrt{1-\rho^2} \, dW_t^{d\perp}$$

$$= -\rho\eta \, dt + dB_t^d$$

i) $\varphi = (V_T - K)_+ \, X_T$

$$P_0 \overset{?}{=} X_0 \left(V_0 \, \Phi(d_+) - K e^{-r^f T} \, \Phi(d_-) \right)$$

$$d_\pm = \frac{\ln(V_0/K) + (r^f \pm \frac{1}{2}\sigma^2) T}{\sigma \sqrt{T}}$$

$$P_0 = e^{-r^d T} \, \mathbb{E}^{\mathbb{Q}^d}\left[(V_T - K)_+ \, X_T \right]$$

define $Z_t = X_t M^F_t$ is a domestic asset. > 0 a.s.

use it as a numeraire asset.

$$\Rightarrow \frac{P_0}{Z_0} = \mathbb{E}^{\mathbb{Q}^Z}\left[\frac{(V_T - K)_+ \, X_T}{\underbrace{Z_T}_{X_T e^{r^f T}}} \right]$$

$$= e^{-r^f T} \, \mathbb{E}^{\mathbb{Q}^Z}\left[(V_T - K)_+ \right]$$

know that $\dfrac{dV_t}{V_t} = (r^F - \sigma \rho \eta) \, dt + \sigma \, dB^d_t$

and $\dfrac{dZ_t}{Z_t} = r^d \, dt + \eta \, dW^d_t$

$$\Rightarrow dW^Z_t = -\eta \, dt + dW^d_t$$

$$dB^Z_t = -\rho \eta \, dt + dB^d_t$$

so $\Rightarrow \dfrac{dV_t}{V_t} = r^F \, dt + \sigma \, dB^Z_t$

ii) $\varphi = (V_T - K)_+ x$

$$P_0 = e^{-r^d T}\, \mathbb{E}^{\mathbb{Q}^d}\left[(V_T - K)_+ x \right]$$

recall $\dfrac{dV_t}{V_t} = (r^f - \rho\sigma\eta)\,dt + \sigma\, dB_t^d$

$$P_0 = x\, e^{-r^d T} \cdot e^{(r^f - \rho\sigma\eta)T}\left(V_0\, \Phi(d_+) - K\, e^{-(r^f - \rho\sigma\eta)T}\, \Phi(d_-) \right)$$

$$d_\pm = \frac{\ln(V_0/K) + (r^f - \rho\sigma\eta \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

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domestic \$

iii) $\quad Q = (V_T X_T - K)_+$

$$P_0 = e^{-r^d T} \mathbb{E}^{\mathbb{Q}^d}\left[(V_T X_T - K)_+ \right]$$

$$\frac{dV_t}{V_t} = (r^F - \rho\sigma\eta)\,dt + \sigma\, dB_t^d$$

$$\frac{dX_t}{X_t} = (r^d - r^F)\,dt + \eta\, dW_t^d$$

$$\frac{d(V_t X_t)}{V_t X_t} = r^d\, dt + \underbrace{\sigma\, dB_t^d + \eta\, dW_t^d}_{(\sigma^2 + \eta^2 + 2\sigma\rho\eta)^{1/2}\, dL_t^d}$$

$\hookrightarrow \bar{\sigma}$

$$P_0 = V_0 X_0\, \Phi(d_+) - K e^{-r^d T}\, \Phi(d_-)$$

$$d_\pm = \frac{\ln\left(\frac{V_0 X_0}{K}\right) + \left(r^d + \frac{1}{2}\bar{\sigma}^2\right) T}{\bar{\sigma}\sqrt{T}}$$

iv) $\quad Q = \left(V_T^{(1)} - V_T^{(2)}\right)_+ x \quad ?$

v) $\quad$ stoch. ir.

$$P_0 = \mathbb{E}^{\mathbb{Q}^d}\left[e^{-\int_0^T r_s^d\, ds} (V_T - K)_+ x \right]$$

$$= B^d_0(T)\ \mathbb{E}^{Q^d}\left[\frac{(V_T - K)_+ \, x}{B^d_T(T)}\right]$$

$$\frac{V_T}{B^d_T(T)}, \quad Z_t = \frac{V_t}{B^d_t(T)}$$

!! is NOT a Q^d-mtg
b/c V_t is not a domestic asset price !!