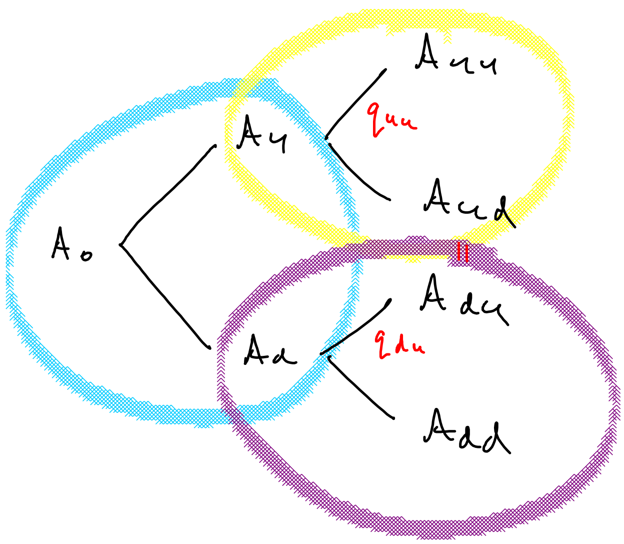
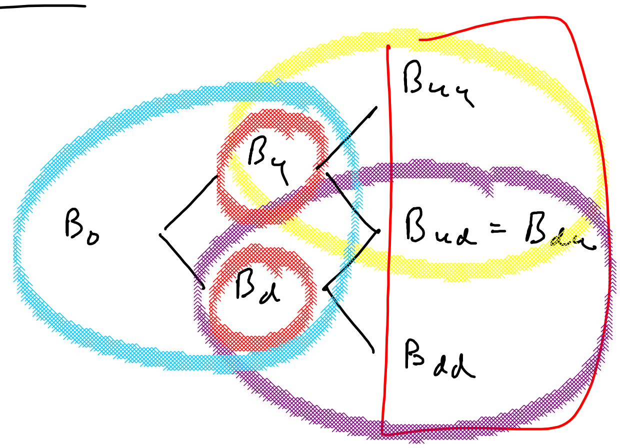
$\underline{r = 0}$





$$A_u = q_{uu} A_{uu} + (1-q_{uu}) A_{ud} \Rightarrow q_{uu}$$

$$A_d = q_{du} A_{du} + (1-q_{du}) A_{dd} \Rightarrow q_{du}$$

$$B_d = q_{du} B_{du} + (1-q_{du}) B_{dd}$$

if $B_T = \varphi(A_T)$ then B can be interpreted as an European option on A.

S_{t_0}
S_{t_1}
$\vdots$
$S_{t_n = 0}$

"match mean + var of data"

↳ returns: $\bar{r}_{t_k} = \dfrac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}$

$$\hat{r}_{t_k} = \ln\left(S_{t_k} / S_{t_{k-1}}\right)$$

$$S_{t_k} = S_{t_{k-1}} \, e^{Z} \qquad , Z \sim N(a; b^2)$$

$$\mathbb{E}[\bar{r}_{t_k}] = \mathbb{E}(e^Z - 1) = e^{a + \frac{1}{2}b^2} - 1 \sim a + \underline{\underline{\tfrac{1}{2}b^2}}$$

$$\mathbb{E}[\hat{r}_{t_k}] = \mathbb{E}[Z] = a$$

↳ convexity correction

$$\mathbb{E}[e^{uX}] = e^{u^2/2}$$

$$X \sim N(0; 1) \qquad Z \stackrel{d}{=} a + bX$$

suppose historically we find

$$\mathbb{E}[\hat{r}] = (\mu - \tfrac{1}{2}\sigma^2)\,\Delta t$$

$$\mathbb{V}[\hat{r}] = (\sigma^2)\,\Delta t$$

$$A_k \begin{cases} A_{k,u} = A_k \, e^{c} \\ A_{k,d} = A_k \, e^{-c} \end{cases}$$

$$A_{k+1} = A_k \, e^{x_k}$$

$x_1, x_2, \ldots$ iid Bernoulli

$\mathbb{P}(x_1 = +c) = p$

$\mathbb{P}(x_2 = -c) = 1-p$

$\Rightarrow$ ln-return are independent

① $\mathbb{E}[\, \hat{r}\,] = \mathbb{E}[\, x_k] = c\, p - c\,(1-p)$

$= c\,(2p-1)$ $= \Delta t \,(\mu - \frac{1}{2}\sigma^2)$

② $\mathbb{V}[\,\hat{r}\,] = \mathbb{V}[\, x_k] = \mathbb{E}[\, x_k^2\,] - (\mathbb{E}[\, x_k])^2$

$= c^2 - c^2\,(2p-1)^2$

$= c^2\,[\, 1 - (2p-1)^2 \,]$ $= \Delta t\, \sigma^2$

① $\Rightarrow$ $p = \frac{1}{2}\left[1 + \frac{\mu - \frac{1}{2}\sigma^2}{c}\, \Delta t \right]$

② $\Rightarrow$ $c^2 \left[1 - \frac{(\mu - \frac{1}{2}\sigma^2)^2}{c^2}\, \Delta t^2 \right] = \Delta t\, \sigma^2$

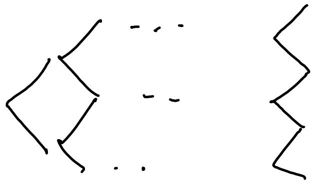
$\Rightarrow$ $c^2 - (\mu - \frac{1}{2}\sigma^2)^2\, \Delta t^2 = \Delta t\, \sigma^2$

$\Rightarrow$ $c = \left(\Delta t\, \sigma^2 + (\mu - \frac{1}{2}\sigma^2)^2\, \Delta t^2 \right)^{1/2}$

$= \sqrt{\Delta t}\;\sigma \sqrt{1 + \left(\frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\right)^2 \Delta t}$

$\sim \sigma\sqrt{\Delta t} + o(\Delta t)$

& $p \sim \frac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \cdot \sqrt{\Delta t} \right) + o(\Delta t)$



$0 \qquad \Delta t \downarrow 0 \qquad T < +\infty$

$$A_T \stackrel{d}{=} \,?$$

$$A_{k+1} = A_k \, e^{x_k}$$
$$= A_k \, e^{\sigma\sqrt{\Delta t}\; y_k}$$

y_k is ± 1 Bernoulli; $\mathbb{P}(y_k = +1) = p$, $\mathbb{P}(y_k = -1) = 1-p$

$$A_T = A_0 \exp\Big\{ \underbrace{\sigma\sqrt{\Delta t} \sum_{k=1}^{N} y_k}_{} \Big\} \qquad \Delta t = \frac{T}{N}$$

by CLT $\quad X \xrightarrow[\Delta t \downarrow 0]{d} \mathcal{N}(\text{mean}, \text{var})$

$$\mathbb{E}[X] = \sigma\sqrt{\Delta t}\; N \,.\, \mathbb{E}[y_1]$$
$$= \sigma\sqrt{\frac{T}{N}} \,.\, N \,.\, (2p-1)$$
$$= \sigma\sqrt{\frac{T}{N}} \,.\, N \,.\left[\frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \,.\, \sqrt{\frac{T}{N}} + o\left(\frac{T}{N}\right) \right]$$

$$\xrightarrow[N \uparrow +\infty]{} T\left(\mu - \tfrac{1}{2}\sigma^2\right)$$

$$\mathbb{V}[X] = \sigma^2 \Delta t \,.\, N \,.\, \mathbb{V}[y_1]$$
$$= \sigma^2 \frac{T}{N} \,.\, N \,.\left[1 - (2p-1)^2 \right]$$

$\quad \hookrightarrow \; 1 - \left(\frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\right)^2 \frac{T}{N}$

$$\xrightarrow[N \uparrow +\infty]{} \sigma^2 T$$

$$\therefore \quad \boxed{\begin{gathered} A_T \overset{d}{=} A_0\, e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z} \\ Z \underset{\mathbb{P}}{\sim} \mathcal{N}(0,1) \end{gathered}}$$

$$\mathbb{E}^{\mathbb{P}}[A_T] = A_0\, e^{(\mu - \frac{1}{2}\sigma^2)T}\, \underbrace{\mathbb{E}^{\mathbb{P}}[e^{\sigma\sqrt{T} Z}]}_{= e^{\frac{1}{2}\sigma^2 T}}$$

$$= A_0\, e^{\mu T}$$

A is log-normal!

"Black-Scholes Model"

$$A \begin{array}{l} \nearrow A\, e^{\sigma\sqrt{\Delta t}} \\ \searrow A\, e^{-\sigma\sqrt{\Delta t}} \end{array}$$

CRR
Cox, Ross, Rubenstein

$$A \begin{array}{l} \nearrow A\, e^{\sigma\sqrt{\Delta t} + \overbrace{(\mu - \frac{1}{2}\sigma^2)}^{\alpha - \frac{1}{2}\sigma^2}\Delta t} \\ \searrow A\, e^{-\sigma\sqrt{\Delta t} + \underbrace{(\mu - \frac{1}{2}\sigma^2)}_{\alpha - \frac{1}{2}\sigma^2}\Delta t} \end{array}$$

$$p = \frac{1}{2}\left[1 + \frac{\mu - \alpha}{\sigma}\sqrt{\Delta t}\right] + o(\Delta t)$$

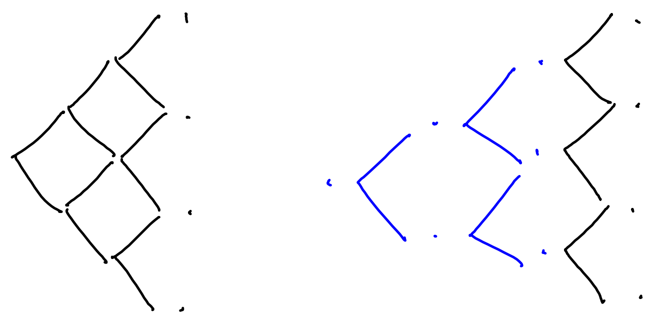
calibrated the model to data, have r & σ ...

$$A \begin{cases} A\, e^{\sigma\sqrt{\Delta t}} \\ A\, e^{-\sigma\sqrt{\Delta t}} \end{cases}$$

$(A-K)_+$
$= \max(A-K, 0)$

value a call on A ...

A :



$$A_k \begin{cases} A_k\, e^{\sigma\sqrt{\Delta t}} \\ A_k\, e^{-\sigma\sqrt{\Delta t}} \end{cases}$$

$$A_k = e^{-r\Delta t}\left[\, q\, A_k\, e^{\sigma\sqrt{\Delta t}} + (1-q)\, A_k\, e^{-\sigma\sqrt{\Delta t}} \right]$$

$$\Rightarrow \quad e^{r\Delta t} = q\left(e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}\right) + e^{-\sigma\sqrt{\Delta t}}$$

$$\Rightarrow \quad \boxed{q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}}$$

all q's are the same

$$\sim \frac{(1+r\Delta t) - \left(1 - \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2\Delta t\right) + o(\Delta t)}{\left(1+\sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2\Delta t\right) - \left(1 - \sigma\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2\Delta t\right) + o(\Delta t)}$$

$$= \frac{\sigma\sqrt{\Delta t} + \left(r - \tfrac{1}{2}\sigma^2\right)\Delta t}{2\,\sigma\sqrt{\Delta t}} + o(\Delta t)$$

$$= \frac{1}{2}\left[1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma} \cdot \sqrt{\Delta t}\right] + \cdots$$

recall $\quad p = \frac{1}{2}\left[1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t}\right] + \cdots$

$$A_T \underset{\mathbb{Q}}{\overset{d}{=}} ?$$

$$A_T = A_0 \exp\Big\{ \underbrace{\sigma\sqrt{\Delta t} \sum_{n=1}^{N} y_n}_{X} \Big\}$$

by CLT $\quad X \underset{\mathbb{Q}}{\sim} \mathcal{N}(\cdot\,;\cdot) \quad$ as $N \uparrow +\infty$

$$\mathbb{E}^{\mathbb{Q}}[X] = \sigma\sqrt{\frac{T}{N}} \cdot N \cdot \mathbb{E}^{\mathbb{Q}}[y_1]$$

$$= \sigma\sqrt{\frac{T}{N}} \cdot N \cdot (2q-1)$$

$$= \sigma\sqrt{\frac{T}{N}} \cdot N \cdot \frac{r - \frac{1}{2}\sigma^2}{\sigma} \cdot \sqrt{\frac{T}{N}} + \cdots$$

$$\xrightarrow[N\uparrow+\infty]{} (r - \tfrac{1}{2}\sigma^2)\, T$$

$$\mathbb{V}^{\mathbb{Q}}[X] = \sigma^2 \Delta t \cdot N \cdot \mathbb{V}[y_1]$$

$$= \sigma^2 \frac{T}{N} \cdot N \cdot \left(1 - (2q-1)^2\right)$$

$$\hookrightarrow \quad 1 - \left(\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right)^2 \cdot \frac{T}{N}$$

$$\xrightarrow[N\uparrow+\infty]{} \sigma^2\, T$$

so $\boxed{A_T \overset{d}{=} A_0 \, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z} \qquad Z \underset{\mathbb{Q}}{\sim} N(0,1)}$

$$\mathbb{E}^{\mathbb{Q}}[A_T] = e^{rT} A_0$$

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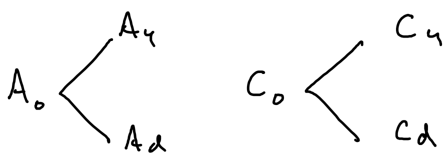
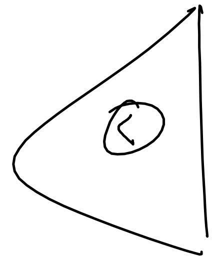
American style derivatives.

exercise allowed at any time up to and including maturity T.

call





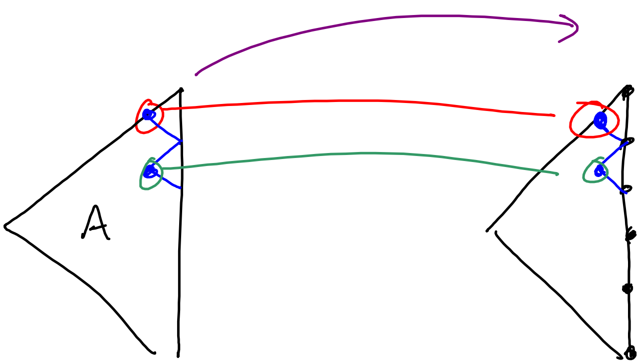
$(K - A_0)_+$

if exercise $\quad C_0 = (A_0 - K)_+$

if don't exercise, receive random outcome C_u or C_d @ next time step.

$$C_0 = e^{-r\Delta t}\,\mathbb{E}^{\mathbb{Q}}[\,C_1\,]$$

$$C_0 = \max\Big(\underbrace{(A_0 - K)_+}_{\substack{\text{intrinsic}\\ \text{or exercise value}}}\overset{(K-A_0)_+}{}\ ;\ \underbrace{e^{-r\Delta t}\,\mathbb{E}^{\mathbb{Q}}[C_1]}_{\substack{\text{hold on}\\ \text{continuation}}}\Big)$$

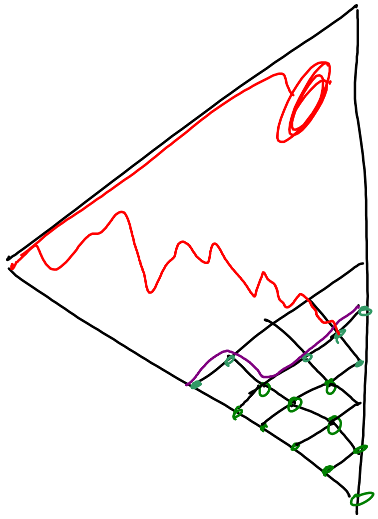


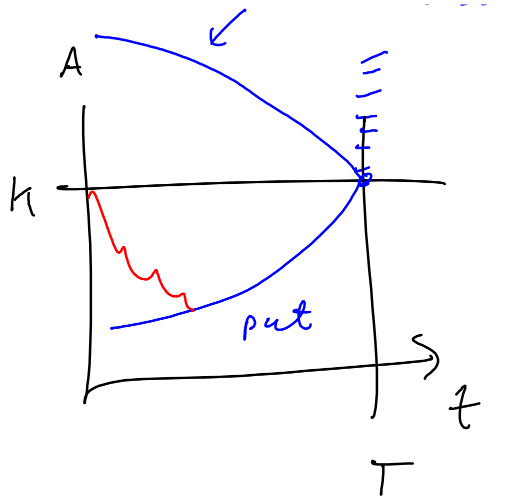
$(A_T - K)_+$

$(K - A_T)_+$

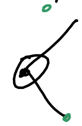
call

exercise bd





$$C_0 = \max(E_0, H)$$



$$E_{1d} > E_0$$

$$H = \frac{1}{1+r}\left(E_{1d}(1-q) + C_{1u}(q)\right)$$

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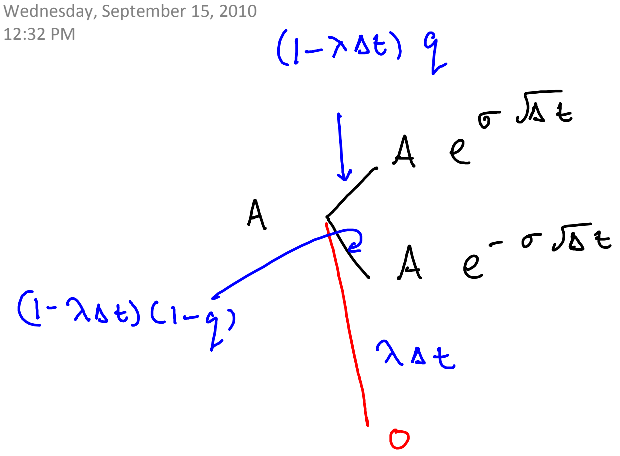

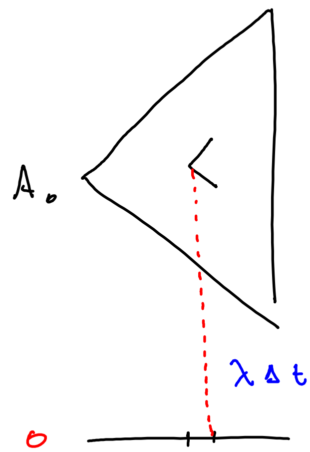

τ = default time

if $\tau \underset{\mathbb{Q}}{\sim}$ exp hazard rate λ

$$f_\tau(s) = \lambda e^{-\lambda s}$$

$$F_\tau(s) = 1 - e^{-\lambda s}$$

$$\mathbb{Q}(\tau \in (t_k, t_{k+1}] \mid \tau > t_k)$$

$$= \mathbb{Q}(\tau \in (0, \Delta t_k])$$

$$= 1 - e^{-\lambda\Delta t_k} \sim \lambda\Delta t_k$$

Find $\mathbb{Q}$!

$$A = e^{-r\Delta t}\Big(\underbrace{A e^{-\lambda\Delta t}}_{} \; q \; e^{\sigma\sqrt{\Delta t}}$$

$$+ A e^{-\lambda\Delta t} (1-q)\, e^{-\sigma\sqrt{\Delta t}}$$

$$+ (1-e^{-\lambda\Delta t}) \quad 0 \Big)$$

$(1-\lambda\Delta t)$; $\lambda\Delta t$

$$e^{(r+\lambda)\Delta t} = q\, e^{\sigma\sqrt{\Delta t}} + (1-q)\, e^{-\sigma\sqrt{\Delta t}}$$

$$\Rightarrow \quad q = \frac{e^{(r+\lambda)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

$$\sim \frac{1}{2}\left[1 + \frac{(r+\lambda) - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t} \right] + \cdots$$

$$\mathbb{E}^{\mathbb{Q}}\left[A_{t_k} \mid \tau > t_k,\ A_{t_{k-1}} \right] = A_{t_{k-1}}\, e^{(r+\lambda)\Delta t}$$

$r+\lambda$ is the default adjusted risk-free rate.

$$A_T \big|_{\tau > T} \overset{d}{=} \ ? \qquad\qquad A_T \overset{d}{=} \ ?$$

$$A_T \big|_{\tau > T} \overset{d}{=} A_0\, e^{(r+\lambda-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\, Z}$$

$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$A_T \overset{d}{=} A_0\, e^{(r+\lambda-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\, Z}\ \mathbb{1}_{\tau > T}$$

Can you find a way to simulate without using τ !?

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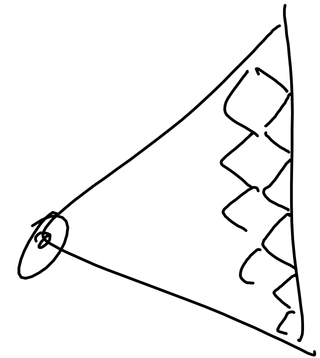
$$A_0 \begin{cases} A_u \\ A_d \end{cases} \qquad C_0 \begin{cases} C_u \\ C_d \end{cases}$$



$$C_0 = \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[C_1]$$

$$\boxed{C_0 = e^{-rT} \mathbb{E}^{\mathbb{Q}}[C_T]}$$

$$C_{n-1} = e^{-r\Delta t} \mathbb{E}^{\mathbb{Q}}[C_n \mid A_{n-1}]$$

$$\begin{aligned} C_{n-2} &= e^{-r\Delta t} \mathbb{E}^{\mathbb{Q}}[C_{n-1} \mid A_{n-2}] \\ &= e^{-r\Delta t} \mathbb{E}^{\mathbb{Q}}[e^{-r\Delta t} \mathbb{E}[C_n \mid A_{n-1}] \mid A_{n-2}] \\ &= e^{-2r\Delta t} \mathbb{E}^{\mathbb{Q}}[C_n \mid A_{n-2}] \end{aligned}$$

$$\mathbb{E}[F(X)] \sim \frac{1}{N} \sum_{n=1}^{N} F(X^{(n)})$$

$X^{(1)}, X^{(2)}, \ldots, X^{(N)}$ are all drawn from dist of X