$$C(t,S) = \mathbb{E}^{\mathbb{Q}}\left[(S_T - K)_+ \right]$$

$$\partial_K C = \mathbb{E}^{\mathbb{Q}}\left[-\mathbb{1}_{S_T > K} \right]$$

$$\partial_{KK} C = \mathbb{E}^{\mathbb{Q}}\left[-D_{S_T, K} \right]$$

$$= -\mathrm{pdf}^{\mathbb{Q}}(S_T = K)$$

Currency (FX)

domestic $\qquad\qquad$ foreign

$r^a \quad (A) \xleftarrow{\;X_t\;} (B) \quad r^b$

$$1 \text{ unit of } \$B = X_t \text{ units of } \$A$$

$$\frac{dX_t}{X_t} = \mu(t, X_t)\, dt + \underbrace{\sigma}_{\sigma(t,X_t)}\, dW_t^{\mathbb{P}}$$

↳ e.g. $\kappa(\theta - X_t)$

- $dB_t^a = r^a B_t^a\, dt$
- $dB_t^b = r^b B_t^b\, dt$
- $(\alpha_t, \beta_t, -1)$ in (B_t^a, B_t^b, g_t)

× claim $g_t = g(t, X_t)$ (domestic (a)), $g_T = G(X_T)$

$$V_t = \alpha_t B_t^a + \beta_t \underbrace{(B_t^b X_t)}_{\text{domestic asset}} - g_t$$

$$V_0 = 0$$

$$dV_t = \alpha_t\, dB_t^a + \beta_t\, d(B_t^b X_t) - dg_t$$

↳ self-financing

$$= \alpha_t r B_t^a\, dt$$
$$+ \beta_t \left(dB_t^b X_t + B_t^b\, dX_t + \underbrace{d[B^b, X]_t}_{0} \right)$$
$$- \left(\mu_t^g g_t\, dt + \sigma_t^g g_t\, dW_t^{\mathbb{P}} \right)$$

do the usual steps ....

$$\begin{cases} \partial_t g + (r^a - r^b)\, x\, \partial_x g + \frac{1}{2}\sigma^2 x^2 \partial_{xx} g = r^a g \\ g(T,x) = G(x) \end{cases}$$

$$g(t,x) = \mathbb{E}^{\mathbb{Q}^a}_{t,x}\left[G(X_T)\, e^{-r^a(T-t)} \right]$$

$$dX_t = (r^a - r^b)\, X_t\, dt + \sigma X_t\, dW_t^{\mathbb{Q}^a}$$

$$\mathbb{E}^{\mathbb{Q}}[\ \cdot \mid X_t = x]$$

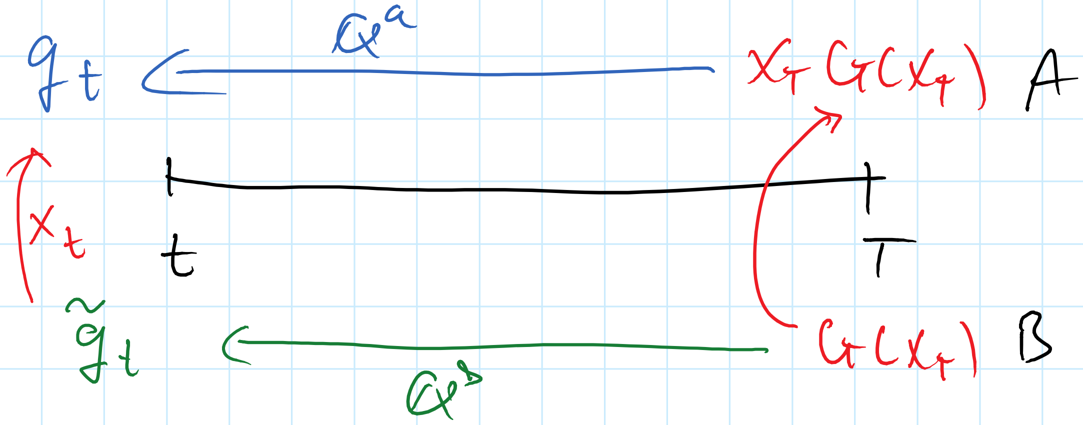
$\mathbb{Q}^a$ is the domestic risk-neutral measure

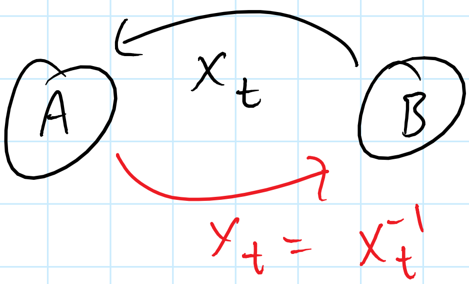
Suppose g_t is a foreign claim

$g_T = G(X_T)$ in foreign dollars.

$$g(t,x) = \mathbb{E}_t^{Q^a}\left[e^{-r^a(T-t)} X_T \, G(X_T) \right]$$

$$\neq X_t \, \mathbb{E}_t^{Q^a}\left[e^{-r^a(T-t)} G(X_T) \right]$$





$$\cancel{dX_t = (r^d - r^a) X_t \, dt + \sigma X_t \, dW_t^{Q^d}}$$

$$dY_t = (r^d - r^a) Y_t \, dt - \sigma Y_t \, dW_t^{Q^d}$$

$$\tilde{g}(t,x) = \mathbb{E}_{t,x}^{Q^d}\left[e^{-r^d(T-t)} \cdot G(X_T) \right]$$

$$G(X_T) = \tilde{G}(Y_T) = G(1/Y_T)$$

to see that $-\sigma Y_t \, dW_t^{Q^d}$ is correct...

to see that $-\sigma Y_t \, dW_t^{\mathbb{Q}^\Delta}$ is correct...

$f(x) = x^{-1}$

$$dY_t = d f(X_t)$$

$$\text{"="} \ \partial_x f \, dX_t + \tfrac{1}{2} \partial_{xx} f \, (dX_t)^2$$

$$= -X_t^{-2} \left(\mu(t, X_t) X_t \, dt + \sigma X_t \, dW_t^{\mathbb{P}} \right)$$

$$+ \tfrac{1}{2} (+2 X_t^{-3}) \cdot \sigma^2 X_t^2 \, dt$$

$$= Y_t \Big((\sigma^2 - \mu(t, 1/Y_t)) \, dt$$

$$- \sigma \, dW_t^{\mathbb{P}} \Big)$$

$$= Y_t \left(\mu_t^Y \, dt - \sigma \, dW_t^{\mathbb{P}} \right)$$

$$dY_t = (r^\Delta - r^a) Y_t \, dt - \sigma Y_t \, dW_t^{\mathbb{Q}^\Delta}$$

$$\Rightarrow dX_t = \left(\sigma^2 - (r^\Delta - r^a) \right) X_t \, dt + \sigma X_t \, dW_t^{\mathbb{Q}^\Delta}$$

$$dX_t = \left((r^a - r^\Delta) + \sigma^2 \right) X_t \, dt + \sigma X_t \, dW_t^{\mathbb{Q}^\Delta}$$

$$= (r^a - r^\Delta) X_t \, dt + \sigma X_t \, dW_t^{\mathbb{Q}^a}$$

So finally,

$$g(t,x) = x\,\tilde{g}(t,x)$$
$$= x\,\mathbb{E}^{\mathbb{Q}^b}_{t,x}\left[e^{-r^b(T-t)} G(X_T)\right]$$
$$= \mathbb{E}^{\mathbb{Q}^a}_{t,x}\left[e^{-r^a(T-t)} G(X_T)\,X_T\right]$$

suppose $G(x) = 1$, i.e. we receive #1B @ T.

clearly, $\tilde{g}(t,x) = \mathbb{E}^{\mathbb{Q}^b}_{t,x}\left[e^{-r^b\tau} \cdot G(X_T)\right]$
$$= e^{-r^b\tau}$$

and $g(t,x) = \mathbb{E}^{\mathbb{Q}^a}_{t,x}\left[e^{-r^a\tau} \cdot X_T\right]$

recall that $dX_t = (r^a - r^b)X_t\,dt + \sigma X_t\,dW_t^{\mathbb{Q}^a}$

$$\Rightarrow X_T = X_t \exp\left\{\left((r^a - r^b) - \tfrac{1}{2}\sigma^2\right)\tau + \sigma(W_T^a - W_t^a)\right\}$$

$$\Rightarrow \mathbb{E}^{\mathbb{Q}^a}_{t,x}[X_T] = x\,e^{((r^a-r^b)-\frac{1}{2}\sigma^2)\tau} \cdot \mathbb{E}\left[e^{\sigma(W_T^a - W_t^a)}\right]$$

$\hookrightarrow e^{\frac{1}{2}\sigma^2\tau}$

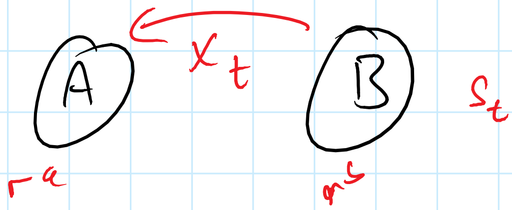
$W_T^a - W_t^a \sim \mathcal{N}(0; T-t)$

$$\Rightarrow g(t,x) = x\,e^{-r^b\tau} = x\,\tilde{g}(t,x)$$

try $G(X_T) = \mathbb{I}_{X_T > K}$

$(X_T - K)_+$

29 October 2014 15:56



$$dS_t = S_t(\mu\, dt + \sigma\, dW_t^s)$$

$$dX_t = X_t(\nu\, dt + \eta\, dW_t^x)$$

W_t^x, W_t^s are correlated ρ.

$$dS_t = S_t(r^b\, dt + \sigma\, dW_t^{bs})$$

$$= S_t(?\, dt + \sigma\, dW_t^{as})$$

note: $\tilde{S}_t = S_t X_t$ is a domesticized foreign asset.

therefore:

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = r^a\, dt + (\quad)\, dW_t^{ax} + (\quad)\, dW_t^{as}$$

then we can find:

$$\frac{dS_t}{S_t} = (\,?\,)\, dt + (0)\, dW_t^{ax} + (\sigma)\, dW_t^{as}$$

$\sigma^2 - r^a$

$r^b - r^a$

$r^b - r^a + \sigma^2$

$$d\tilde{S}_t = d(S_t X_t)$$

$$= dS_t\, X_t + S_t\, dX_t + d[S,X]_t$$

$$= S_t X_t (\mu\, dt + \sigma\, dW_t^S)$$
$$+ S_t X_t (\nu\, dt + \eta\, dW_t^X)$$
$$+ S_t X_t \rho \sigma \eta\, dt$$

$$\Rightarrow \quad \frac{d\tilde{S}_t}{\tilde{S}_t} = (\mu + \nu + \sigma\eta\rho)\, dt + \sigma\, dW_t^S + \eta\, dW_t^X$$

write:

$$dW_t^{aS} = \lambda_t^{aS}\, dt + dW_t^S$$
$$dW_t^{aX} = \lambda_t^{aX}\, dt + dW_t^X$$

$$\Rightarrow \frac{d\tilde{S}_t}{\tilde{S}_t} = (\mu + \nu + \sigma\eta\rho - \lambda_t^{aS}\sigma - \lambda_t^{aX}\eta)\, dt + \sigma\, dW_t^{aS} + \eta\, dW_t^{aX}$$

r^a ①

domesticize Foreign bank account:

$\tilde{B}_t^D = X_t B_t^D$ is a "domestic asset".

$$\Rightarrow d\tilde{B}_t^D = dX_t\, B_t^D + X_t\, dB_t^D + d[X, B^D]_t$$

$\overset{\curvearrowright 0}{}$

$$= (\nu\, dt + \eta\, dW_t^X)\, X_t B_t^D$$
$$+ X_t B_t^D\, r^D\, dt$$

$$\Rightarrow \frac{d\tilde{B}_t^D}{\tilde{B}_t^D} = (\nu + r^D)\, dt + \eta\, dW_t^X$$
$$= (\nu + r^D - \lambda_t^X \eta)\, dt + \eta\, dW_t^{aX}$$

$$\frac{\quad}{\tilde{B}^{\triangle}_t} = (\nu + r^{\triangle} - \lambda^{x}_t \eta)\,dt + \eta\, dW^{aX}_t$$

$\hookrightarrow r^a$ ②

$$② \Rightarrow \boxed{\lambda^{aX}_t = \frac{\nu + r^{\triangle} - r^a}{\eta}}$$

$$① \Rightarrow \mu + \nu + \sigma\eta\rho - \lambda^{as}_t \sigma - \lambda^{aX}_t \eta = r^a$$

$$\Rightarrow \mu + \sigma\eta\rho + r^a - r^{\triangle} - \lambda^{as}_t \sigma = r^a$$

$$\Rightarrow \boxed{\lambda^{as}_t = \frac{\mu - r^{\triangle} + \sigma\eta\rho}{\sigma}}$$

Hence we have that

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW^s_t$$

$$= (\mu - \sigma\lambda^{as}_t)\,dt + \sigma\, dW^{as}_t$$

$$\boxed{\frac{dS_t}{S_t} = (r^{\triangle} - \sigma\eta\rho)\,dt + \sigma\, dW^{as}_t}$$

$$\& \quad \frac{dX_t}{X_t} = \nu\, dt + \eta\, dW^X_t$$

$$= (\nu - \eta\lambda^{aX}_t)\,dt + \eta\, dW^{aX}_t$$

$$\boxed{\frac{dX_t}{X_t} = (r^a - r^{\triangle})\,dt + \eta\, dW^{aX}_t}$$

$$\boxed{\frac{dX_t}{X_t} = (r^a - r^b)\,dt + \eta\, dW_t^{a\,X}}$$

if g_t is a claim paying $G(X_T, S_T)$ in

a) domestic dollars

b) Foreign dollars

a) $$g(t,x,s) = \mathbb{E}^{\mathbb{Q}^a}_{t,x,s}\left[e^{-r^a \tau}\, G(X_T, S_T)\right]$$

$$= x\, \mathbb{E}^{\mathbb{Q}^b}_{t,x,s}\left[e^{-r^b \tau}\, \frac{G(X_T, S_T)}{X_T}\right]$$

b) $$g(t,x,s) = x\, \mathbb{E}^{\mathbb{Q}^b}_{t,x,s}\left[e^{-r^b \tau}\, G(X_T, S_T)\right]$$

$$= \mathbb{E}^{\mathbb{Q}^a}_{t,x,s}\left[e^{-r^a \tau}\, G(X_T, S_T) \cdot X_T\right]$$

$$\Big(\underbrace{0.99^{-1}}_{\text{fixed FX}} \cdot \underbrace{S_T}_{\text{US}} - \underbrace{50}_{\text{CAD}}\Big)_+$$

$$\big(\underset{\$B \to \$A}{a}\ \underset{\$B}{S_T} - \underset{\$A}{K}\big)_+ \qquad \text{quanto call}$$

$$\big(\underbrace{X_T S_T}_{\$A} - \underbrace{K}_{\$A}\big)_+$$

$$\big(\underbrace{a S_T}_{\$A} - \underbrace{K X_T}_{\$A}\big)_+$$

suppose $r^a = r^b = 0$

$$\mathbb{E}^{\mathbb{Q}^a}\big[(a S_T - K X_T)_+\big]$$

$$= \mathbb{E}^{\mathbb{Q}^a}\big[(a S_T - K X_T)\, \mathbb{1}_{a S_T > K X_T}\big]$$

$$= \mathbb{E}^{\mathbb{Q}^a}\big[a S_T\, \mathbb{1}_{a S_T > K X_T} - K X_T\, \mathbb{1}_{a S_T > K X_T}\big]$$

$$\mathbb{E}^{\mathbb{Q}^a}\left[\left(a \frac{S_T}{X_T} - K\right)_+ X_T\right]$$

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}^a} = \frac{X_T}{\mathbb{E}^{\mathbb{Q}^a}[X_T]}$$

$$\frac{\quad}{d\,\mathbb{Q}^a} = \frac{\quad}{\mathbb{E}^{\mathbb{Q}^a}[X_T]}$$

$$\mathbb{E}^{\mathbb{Q}^a}[X_T]\;\mathbb{E}^{\mathbb{Q}^*}\left[\left(a\left(\frac{S_T}{X_T}\right)-K\right)_+\right]$$

$$Y_t = (S_t / X_t)$$

$$dY_t = \cdots\, dt + \cdots\, dW_t^{*S} \cdots\, dW^{*X}$$

$$\eta_t = \left.\frac{d\,\mathbb{Q}^*}{d\,\mathbb{Q}^a}\right|_{\mathcal{F}_t} = \frac{\mathbb{E}^{\mathbb{Q}^a}[X_T \mid \mathcal{F}_t]}{\mathbb{E}^{\mathbb{Q}^a}[X_T \mid \mathcal{F}_0]}$$

$$= \exp\left\{\cdots\, t + \cdots\, W_t^{aX}\right\}$$