**UNIVERSITY OF TORONTO**

**Faculty of Arts and Science**

**Term Test, October 20th, 2009**

**ACT 460 / STA 2502**

**DURATION -** 120 **minutes**

EXAMINER: Prof. S. Jaimungal

COURSE CODE (**circle one**): ACT 460 STA 2502

LAST NAME: ____________________________________________

FIRST NAME: ____________________________________________

STUDENT #: ____________________________________________

*Each question is worth 10 points*

– NOT ALL QUESTIONS ARE OF THE SAME DIFFICULTY .

**Please write clearly!**

| 1 *[10]* | 2 *[10]* | 3 *[10]* | 4 *[10]* | 5 *[10]* | 6 *[10]* | Total *[60]* |
|---|---|---|---|---|---|---|
| | | | | | | |

1. *[10]* Please indicate true or false. **no explanations required**

   $-1$ **for incorrect answer,** $+2$ **for correct answer,** $0$ **for blank answer** .

   (a) [T] [F]

   All two period, two state (binomial) economies are arbitrage free.

   (b) [T] [F]

   The price of a call option always decreases with increasing volatility.

   (c) [T] [F]

   If the branching probabilities are unique, then all contingent claims can be replicated.

   (d) [T] [F]

   The risk-neutral return of the short rate of interest in a stochastic interest rate model is equal to $r$.

   (e) [T] [F]

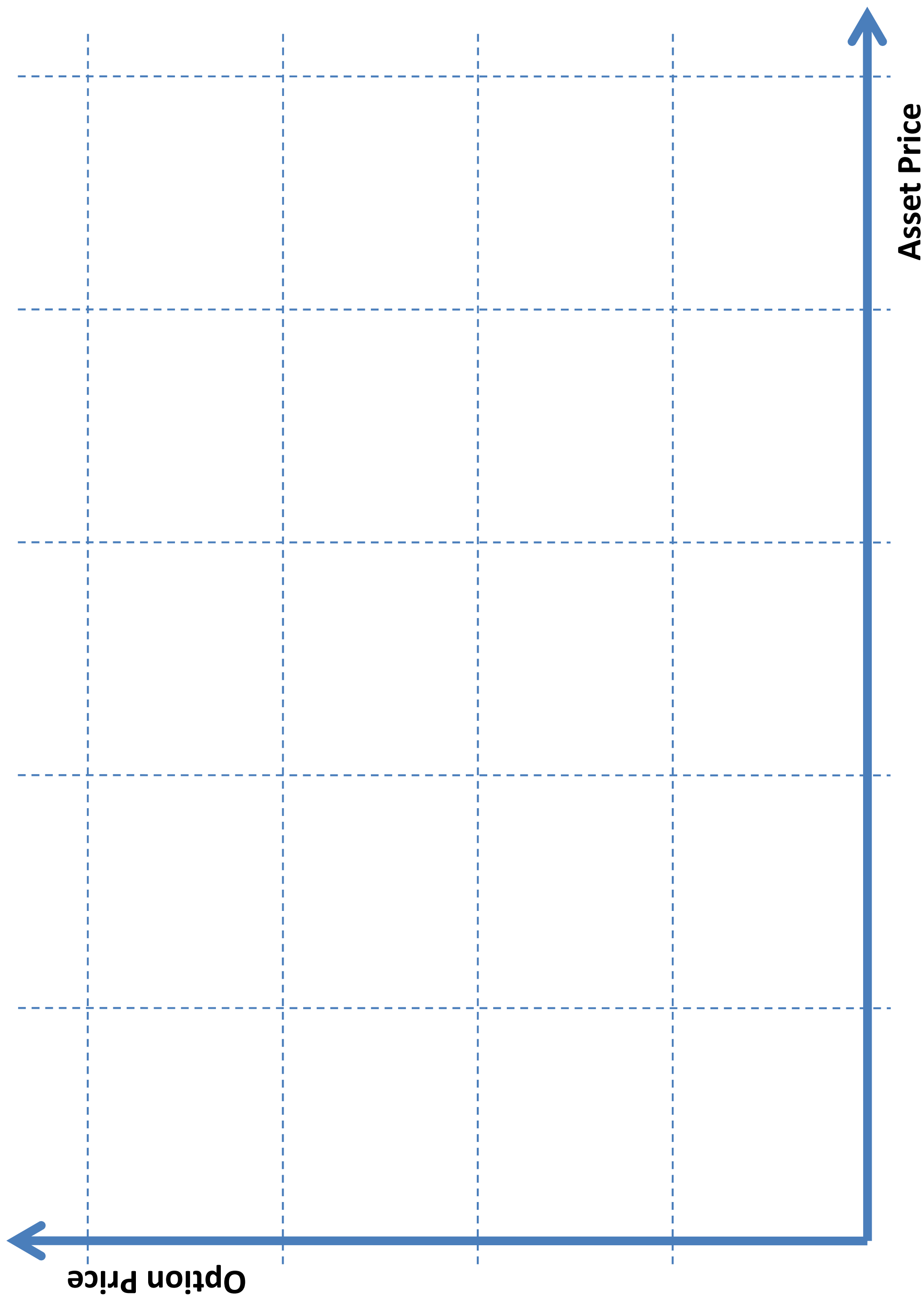
   Suppose interest rates are zero. An at-the-money put option is worth 0.80 and a call option with the same strike and maturity is worth 0.75. This economy admits an arbitrage.

   *[ At-the-money means the strike equals the spot.]*

2. Sketch the option price as a function of the current spot-level for maturities of $T = 0$, $T = 1$ month and $T = 1$ year for

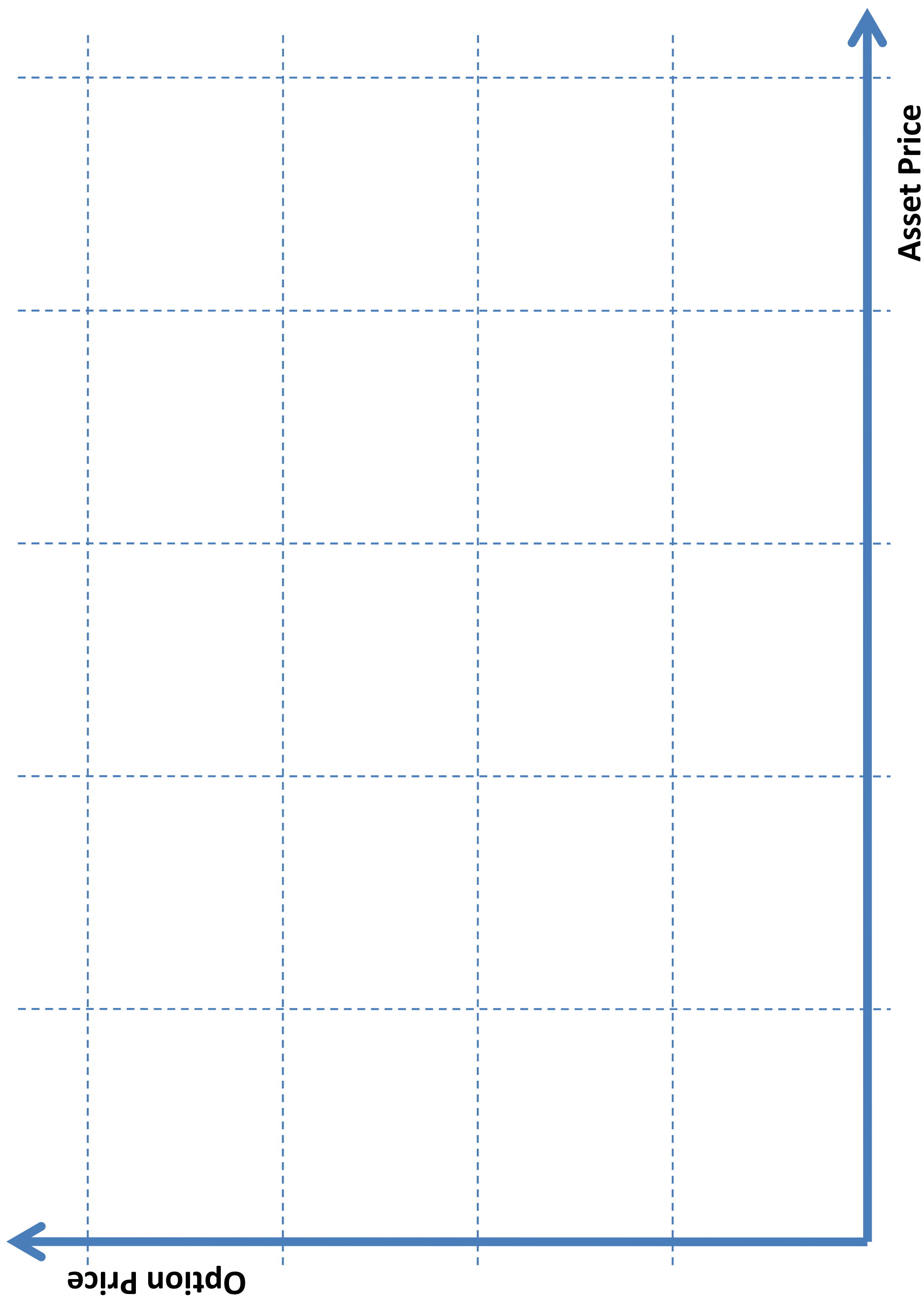
   (a) *[5]* digital call option (which pays 1 if $S > K$ and 0 otherwise).

   [draw the three curves on the same graph, clearly label them and any interesting points.]

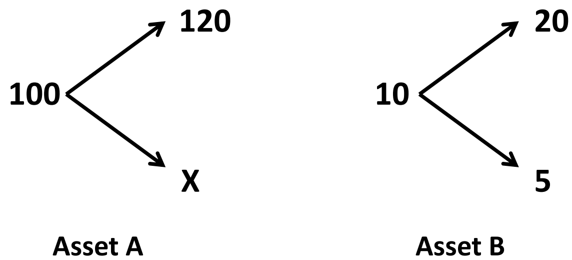


(b) *[5]* A portfolio of 4 long puts and 1 long call, both struck at \$1.

[draw the three curves on the same graph, clearly label them and any interesting points..]



3. *[10]* Consider an economy with the two traded assets below. Find the values of $X$ such that the economy is free of arbitrage.



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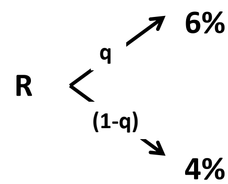
4. Consider the interest rate tree shown in the diagram below – each time step is 1-year. The rates correspond to effective discounting – e.g. discounting over the first period is $1/(1+R)$. The probabilities shown are risk-neutral probabilities.




(a) *[6]* The price of a one-year bond on a notional of \$100 is \$95.2381. As well, a 2-year coupon bearing bond with coupons of \$5 paid every year and notional of \$100 is valued at par. Calibrate this model to the market prices, i.e. determine $R$ and $q$ such that the market prices are equal to the model prices.

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(b) *[4]* Now assume that $q = \frac{1}{2}$ and $R = 5\%$. As well, you can only trade using the 1-year and 2-year **zero coupon** bonds with notionals of \$100 (i.e. 1-year zero coupon bond pays \$100 at year 1, and the 2 year zero coupon bond pays \$100 at year 2).

What is the replication strategy of an option which pays \$100 if the interest rate drops to 4%?

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5. Assume an equity price $S_t$ is modeled as in the Black-Scholes model (i.e. the limiting case of the CRR model as $\Delta t \downarrow 0$ and interest rates are constant at $r$). For each of the following, write your answers terms of $\Phi(x) \triangleq \mathbb{Q}(Z < x)$ where $Z$ is a standard normal random variable under the risk-neutral measure $\mathbb{Q}$.

   (a) *[5]* Derive an expression for the ($t = 0$) price of an option with $T$-maturity payoff

$$\varphi = \min(S_T \; ; \; K) \; .$$

   Here $K$ is a constant.

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(b) *[5]* Derive an expression for the $(t = 0)$ price of a forward start option with $T$-maturity payoff

$$\varphi = \min(S_T\ ;\ k\ S_U)\ .$$

Here, $0 < U < T$ and $k$ is a proportionality constant.

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6. Consider the CRR model of stock prices

$$S_{n\Delta t} = S_{(n-1)\Delta t} \exp\{\sigma\sqrt{\Delta t}\, x_n\}$$

where $x_1, x_2, \ldots$ are iid r.v. with $\mathbb{P}(x_1 = +1) = p$ and $\mathbb{P}(x_1 = -1) = 1 - p$. Interest rates are constant so that the money-market account $M_t$ evolves as

$$M_{n\Delta t} = M_{(n-1)\Delta t} \exp\{r\Delta t\}$$

(a) *[6]* Prove that under the measure induced by using $S$ as a numeraire asset (call this measure $\mathbb{Q}_S$), as $\Delta t \downarrow 0$ one has

$$S_T \stackrel{d}{=} S \exp\{(r + \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z\}$$

where $Z \stackrel{\mathbb{Q}_S}{\sim} \mathcal{N}(0, 1)$.

[Note that the drift is $r + \frac{1}{2}\sigma^2$ and NOT $r - \frac{1}{2}\sigma^2$ as it is under the risk-neutral measure $\mathbb{Q}$.]

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(b) *[4]* Using the measures $\mathbb{Q}_S$ and $\mathbb{Q}$, show that the $(t = 0)$ price of a $T$-maturity put option is

$$K\ e^{-rT}\ \Phi(-d_-) - S\ \Phi(-d_+), \qquad d_\pm = \frac{\ln(S/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\ .$$

**YOU ARE NOT ALLOWED TO COMPUTE INTEGRALS IN THIS QUESTION.**

[Hint: Write the put payoff in terms of a digital option $K\mathbb{I}_{S_T<K}$ and an asset-or-nothing option $S_T\mathbb{I}_{S_T<K}$ and value each separately.]

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