Tuesday, September 30, 2008
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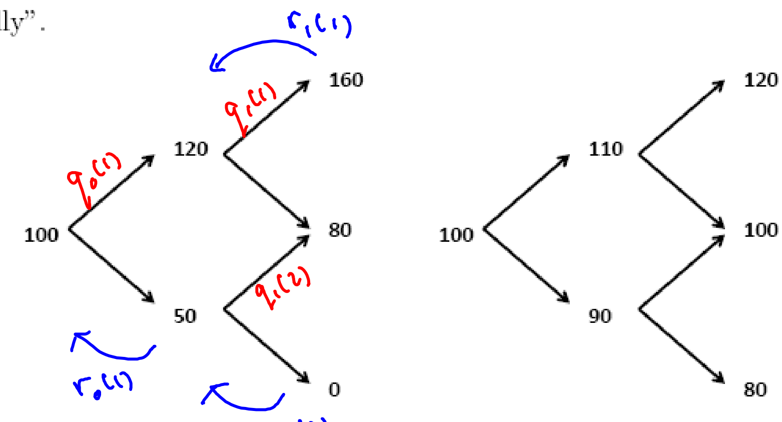
4. The following two assets are being actively traded in a two-period binomial market economy. Asset A behaves like a stock which may default, while asset B behaves "normally".



(a) Asset A (b) Asset B

(a) [5]** Determine all relevant risk-neutral probabilities and short rates of interest.

(b) [5]** Using risk-neutral valuation, compute the price and replication strategy for a two-period American put option on asset A struck at 90.

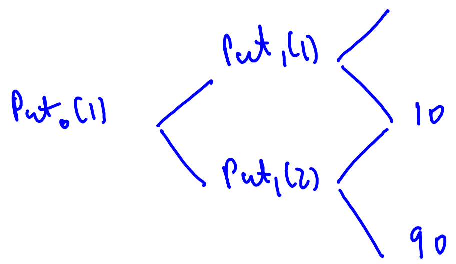
a) In the diagram above I have drawn in the various risk-neutral probabilities and short rates. Using the risk-neutral pricing equations for both assets leads to a linear system at each node:

$$\left.\begin{aligned} 100\,(1+r_0(1)) &= 120\,q_0(1) + 50\,(1-q_0(1)) \\ 100\,(1+r_0(1)) &= 110\,q_0(1) + 90\,(1-q_0(1)) \end{aligned}\right\} \Rightarrow \quad \begin{aligned} r_0(1) &= 3/50 = 6\% \\ q_0(1) &= 4/5 = 80\% \end{aligned}$$

$$\left.\begin{aligned} 120\,(1+r_1(1)) &= 160\,q_1(1) + 80\,(1-q_1(1)) \\ 110\,(1+r_1(1)) &= 120\,q_1(1) + 100\,(1-q_1(1)) \end{aligned}\right\} \Rightarrow \quad \begin{aligned} r_1(1) &= 0 \\ q_1(1) &= \tfrac{1}{2} = 50\% \end{aligned}$$

$$\left.\begin{aligned} 50\,(1+r_1(2)) &= 80\,q_1(2) \\ 90\,(1+r_1(2)) &= 100\,q_1(2) + 80\,(1-q_1(2)) \end{aligned}\right\} \Rightarrow \quad \begin{aligned} r_1(2) &= \tfrac{1}{31} \simeq 3.2\% \\ q_1(2) &= \tfrac{20}{31} \simeq 64.5\% \end{aligned}$$

b) Price tree



$$holdPut_1(1) = (1+r_1(1))^{-1}[10(1-q_1(1))]$$
$$= 5$$

$$\text{Intrinsic } Put_1(1) = (90-120)_+ = 0$$

$$\therefore \quad Put_1(1) = 5$$

$$holdPut_1(2) = (1+r_1(2))^{-1}[10\,q_1(2) + 90(1-q_1(2))]$$
$$= 37.19$$

$$\text{Intrinsic } Put_1(2) = (90-50)_+ = 40$$

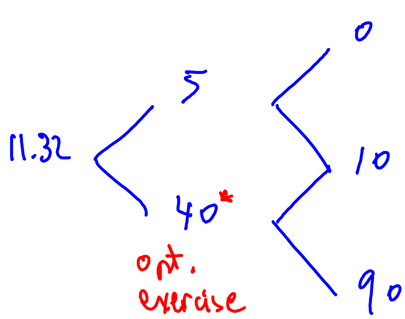
$$\therefore \quad Put_1(2) = 40$$ optimal exercise!

$$hold\ Put_0(1) = (1+r_0(1))^{-1}[5\,q_0(1) + 40(1-q_0(1))]$$
$$= 11.32$$

$$\text{Intrinsic } Put_0(1) = (90-100)_+ = 0$$

$$\therefore \quad Put_0(1) = 11.32$$

replication strategy:



at this node the option will be exercised $\therefore$ we do not "replicate" payoff of 10, 90.

at $t=1$, state $=1$ need to replicate payoff since you will hold option.

$$\alpha_1(1) \times 160 + \beta_1(1) \times 120 = 0$$

$$\alpha_1(1) \times 80 + \beta_1(1) \times 100 = 10$$

$$\Rightarrow \quad \alpha_1(1) = -0.1875 \quad , \quad \beta_1(1) = 0.25$$

at $t=0$, state 1 need to replicate payoff since you will hold option.

$$\alpha_0(1)\ 120 + \beta_0(1)\ 110 = 5$$

$$\alpha_0(1)\ 50 + \beta_0(1)\ 90 = 40$$

$$\Rightarrow \quad \alpha_0(1) = -0.745 \quad , \quad \beta_0(1) = 0.858$$

5. Determine the value (at $t = 0$) of a contingent claim having the following payoff:

   (a) S_T^α

   (b) $\mathbb{I}(S_T > K)$

   (c) **[5]** ** $S_T\, \mathbb{I}(S_T > K)$

   (d) **[5]** ** $((S_T - K)_+)^m$, where m is a strictly positive integer.

   at the maturity date T. Assume that $S_T = S_0\, e^X$ where $X \sim \mathcal{N}\left((\mu - \frac{1}{2}\sigma^2)T,\, \sigma^2 T\right)$ (under the real-world measure $\mathbb{P}$) and the continuous risk-free rate is r.

All options must be valued under the risk-neutral measure. We showed in class that if $S_T = S_0 e^X$ $\quad X \underset{\mathbb{P}}{\sim} \mathcal{N}(\cdot\, ; \sigma^2 T)$

then $\quad X \underset{\mathbb{Q}}{\sim} \mathcal{N}((r - \frac{1}{2}\sigma^2)T;\ \sigma^2 T)$.

$\mathbb{Q}$

c)

$$V_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[\,S_T\,\mathbb{1}(S_T > K)\,]$$

$$= e^{-rT}\int_{-\infty}^{\infty} S_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}z}\,\mathbb{1}_{z>z_*}\, e^{-\frac{1}{2}z^2}\frac{dz}{\sqrt{2\pi}}$$

where $S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}z_*} = K$

$$\Rightarrow z^* = -\frac{\ln(S_0/K) + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\Rightarrow V_0 = S_0\, e^{-\frac{1}{2}\sigma^2 T}\int_{z_*}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2 + \frac{1}{2}\sigma^2 T}\frac{dz}{\sqrt{2\pi}}$$

$$= S_0\,\Phi(-z_* + \sigma\sqrt{T})$$

$$= S_0\,\Phi(d_+)$$

$$d_\pm = \frac{\ln(S_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

d)

$$V_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[\,(S_T - K)_+^m\,]$$

$$= e^{-rT}\sum_{n=0}^{m}\binom{n}{m}\mathbb{E}^{\mathbb{Q}}[\,S_T^n\,K^{m-n}\,\mathbb{1}_{S_T>K}\,]$$

now $\mathbb{E}^{\mathbb{Q}}[\,S_T^n\,\mathbb{1}_{S_T>K}\,]$

$$= \int_{z_*}^{\infty} S_0^{n} \, e^{(r-\frac{1}{2}\sigma^2)nT + n\sigma\sqrt{T}z} \, e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}$$

$$= S_0^{n} e^{(r-\frac{1}{2}\sigma^2)nT} \int_{z_*}^{\infty} e^{-\frac{1}{2}(z - n\sigma\sqrt{T})^2 + \frac{1}{2}n^2\sigma T} \frac{dz}{\sqrt{2\pi}}$$

$$= S_0^{n} e^{(r + \frac{1}{2}\sigma^2(n-1))nT} \, \Phi(-z_* + n\sigma\sqrt{T})$$

$$\therefore V_0 = \sum_{n=0}^{m} \binom{m}{n} K^{m-n} S_0^{n} \, e^{(r+\frac{1}{2}\sigma^2 n)(n-1)T} \, \Phi(-z_* + n\sigma\sqrt{T})$$

6. [5] !! Let $\{t_j : j = 0, \ldots, m\}$ be an ordered series of times $t_0 = 0 < t_1 < t_2 < \ldots < t_m = T$, and let $\{X_i : i = 1, \ldots, m\}$ denote a set of independent normal random variables with means of $\mu\,(t_i - t_{i-1})$ variances of $\sigma^2\,(t_i - t_{i-1})$. Suppose that an asset's price at time t_i are modeled by exponentiation of these r.v.s:

$$S(t_i) = S(t_{i-1})\, e^{X_i}\ . \tag{1}$$

Define $\overline{S}(n)$ as the geometric average of the asset's price over the first n ordered times ($n \le m$). That is, $\overline{S}(n) := \left(\prod_{j=1}^{n} S(t_j)\right)^{1/n}$. Determine the value of a call option written on $\overline{S}(n)$ with strike K maturing at T.
[Hint: What is the distribution of $\overline{S}(n)$?]

$S(t_i) = S(t_{i-1})\ e^{X_i}$

$\Rightarrow\ S(t_i) = S(0)\ \exp\{X_1 + \cdots + X_i\}$

let $Y_i \equiv X_1 + \cdots + X_i$

then $Y_i \sim$ Normal since X_i's are.

$\mathbb{E}[Y_n] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} (t_i - t_{i-1})\ \mu$

$= (t_n - t_0)\ \mu \quad \Rightarrow \boxed{\mathbb{E}[Y_n] = \mu\, t_n}$

$Var[Y_n] = \sum_{i=1}^{n} Var[X_i] = \sum_{i=1}^{n} (t_i - t_{i-1})\ \sigma^2$

X_i's are independent $= (t_n - t_0)\ \sigma^2$

$\Rightarrow \boxed{Var[Y_n] = \sigma^2\, t_n}$

We will require the covariance of Y_i, Y_j $i \neq j$ eventually so compute it now..

$$\mathrm{Covar}[Y_i, Y_j] = \mathrm{Covar}\left[\sum_{p=1}^{i} X_p, \sum_{q=1}^{j} X_q\right]$$
$$= \sum_{p=1}^{i} \sum_{q=1}^{j} \mathrm{Covar}[X_p, X_q]$$

notice that $\mathrm{Covar}[X_p, X_q] = \begin{cases} (t_p - t_{p-1})\sigma^2 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$

$$\therefore \mathrm{Covar}[Y_i, Y_j] = \sum_{p=1}^{\min(i,j)} (t_p - t_{p-1})\sigma^2$$

$$\Rightarrow \boxed{\mathrm{Covar}[Y_i, Y_j] = t_{\min(i,j)}\,\sigma^2}$$

now $\bar{S}(n) = \left(\prod_{j=1}^{n} S(t_j)\right)^{1/n} \Rightarrow \ln \bar{S}(n) = \frac{1}{n}\sum_{j=1}^{n} \ln S(t_j)$

$\Rightarrow \ln \bar{S}(n) = \ln S(0) + \frac{1}{n}\sum_{j=1}^{n} Y_j \Rightarrow \ln \frac{\bar{S}(n)}{S(0)} = \frac{1}{n}\sum_{j=1}^{n} Y_j$

since Y_i's are normal $\Rightarrow \ln \frac{\bar{S}(n)}{S(0)}$ is normal

$\therefore$ $\bar{S}(n)$ is log-normal. Need mean and variance parameters.

$$\mathbb{E}\left[\ln \frac{\bar{S}(n)}{S(0)}\right] = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[Y_j] \Rightarrow \boxed{\mathbb{E}\left[\ln \frac{\bar{S}(n)}{S(0)}\right] = \frac{\mu}{n}\sum_{j=1}^{n} t_j}$$

$$Var\left[\ln \frac{\bar{S}(n)}{S(0)}\right] = \frac{1}{n^2} Var\left(\sum_{j=1}^{n} Y_j\right) = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n} Covar[Y_i, Y_j]$$

$$= \frac{\sigma^2}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n} t_{min(i,j)}$$

$$= \frac{\sigma^2}{n^2}\sum_{j=1}^{n}\left(\sum_{i=1}^{j} t_i + \sum_{i=j+1}^{n} t_j\right)$$

$$= \frac{\sigma^2}{n^2}\left(\sum_{j=1}^{n}\sum_{i=1}^{j} t_i + \sum_{j=1}^{n}(n-j)\,t_j\right)$$

$$= \frac{\sigma^2}{n^2}\left(\sum_{j=1}^{n}(n-j+1)\,t_j + \sum_{j=1}^{n}(n-j)\,t_j\right)$$

$$\Rightarrow \boxed{Var\left[\ln \frac{\bar{S}(n)}{S(0)}\right] = \frac{\sigma^2}{n^2}\sum_{j=1}^{n}\left(2(n-j)+1\right)t_j}$$

Alternatively,

$$Var\left[\ln \frac{\bar{S}(n)}{S(0)}\right] = Var\left[\frac{1}{n}\sum_{j=1}^{n} Y_j\right]$$

$$= \frac{1}{n^2} Var\Big[\begin{aligned}&X_1\\ &+X_1+X_2\\ &+X_1+X_2+X_3\\ &\quad\vdots\\ &+X_1+X_2+X_3+\cdots+X_n\Big]\end{aligned}$$

$$= \frac{1}{n^2} Var[\, nX_1 + (n-1)X_2 + \cdots + X_n]$$

$$= \frac{1}{n^2} Var\left[\sum_{i=1}^{n} (n-i+1) X_i \right]$$

since X_i's are independent

$$= \frac{1}{n^2} \sum_{i=1}^{n} (n-i+1)^2 \, Var[X_i]$$

$$\Rightarrow \boxed{Var\left[\ln \frac{\bar{S}(u)}{S(0)}\right] = \frac{1}{n^2} \sum_{i=1}^{n} (n-i+1)^2 (t_i - t_{i-1}) \sigma^2}$$

This expression looks different from the previous expression, but in fact they are equal! Here's the proof...

$$\sum_{i=1}^{n} (n-i+1)^2 (t_i - t_{i-1})$$

$$= \sum_{i=1}^{n} (n-i+1)^2 t_i - \sum_{i=1}^{n} (n-i+1)^2 t_{i-1}$$

$$= \sum_{i=1}^{n} (n-i+1)^2 t_i - \sum_{i'=0}^{n-1} (n-i')^2 t_{i'}$$ by writing $i-1 = i'$

$$= \sum_{i=1}^{n} (n-i+1)^2 t_i - \sum_{i=1}^{n-1} (n-i)^2 t_i$$ since $t_0 = 0$ + replacing $i' \to i$

$$= \sum_{i=1}^{n-1} \left((n-i+1)^2 - (n-i)^2\right) t_i + (n-n+1)^2 t_n$$

collecting sums

last term of previous line's first term.

$$= \sum_{i=1}^{n-1} (2(n-i)+1)\, t_i \quad + t_n$$

$$= \sum_{i=1}^{n} (2(n-i)+1)\, t_i$$

$$\boxed{\text{To summarize: } \bar{S}(n) = S(0)\, e^{Z(n)} \text{ is lognormal with } \mathbb{E}[Z(n)] = \frac{\mu}{n}\sum_{i=1}^{n} t_i \,; \quad \mathrm{Var}[Z(n)] = \frac{\sigma^2}{n^2}\sum_{i=1}^{n}(2(n-i)+1)\, t_i}$$

For call option price need $\mathbb{Q}$ dynamics..

$$S_{t_j} = S_{t_{j-1}} e^{X_j} \,; \quad X_j \underset{\mathbb{Q}}{\sim} \mathcal{N}\left((r - \tfrac{1}{2}\sigma^2)(t_j - t_{j-1}) ;\ \sigma^2 (t_j - t_{j-1}) \right)$$

so previous result implies

$$\bar{S} = S(0)\, e^{X},$$

$$X \underset{\mathbb{Q}}{\sim} \mathcal{N}\Big(\underbrace{(r - \tfrac{1}{2}\sigma^2) \tfrac{1}{n} \sum_{i=1}^{n} t_i}_{\triangleq (\bar{r} - \frac{1}{2}\bar{\sigma}^2) t_n} ;\ \underbrace{\sigma^2 \tfrac{1}{n^2} \sum (2(n-i)+1) t_i}_{\triangleq \bar{\sigma}^2 t_n} \Big)$$

then call price

$$C_0 = e^{-r t_n} \mathbb{E}^{\mathbb{Q}}\left[(\bar{S}_T - K)_+ \right]$$

$$= e^{(\bar{r} - r) t_n}$$

$$\times e^{-\bar{r} t_n} \mathbb{E}^{\mathbb{Q}}\left[\left(S_0 e^{(\bar{r} - \frac{1}{2}\bar{\sigma}^2) t_n + \bar{\sigma}\sqrt{t_n}\, z} - K \right)_+ \right]$$

where $z \underset{Q}{\sim} \mathcal{N}(0,1)$

$$\therefore \quad c_0 = e^{(\bar{r}-r)t_n}\left(S_0 \Phi(d_+) - K e^{-\bar{r} t_n} \Phi(d_-)\right)$$

$$= e^{(\bar{r}-r)t_n} S_0 \Phi(d_+) - K e^{-r t_n} \Phi(d_-)$$

$$d_\pm = \frac{\ln(S_0/K) + (\bar{r} \pm \frac{1}{2}\bar{\sigma}^2)t_n}{\bar{\sigma}\sqrt{t_n}}$$

similar to Black-Scholes with adjusted rate, spot, and vol.