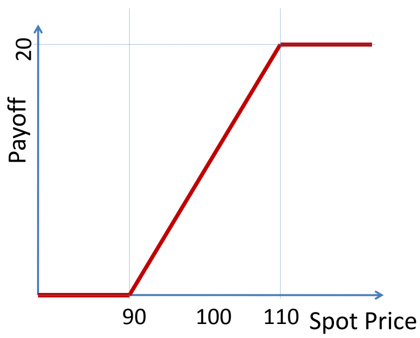
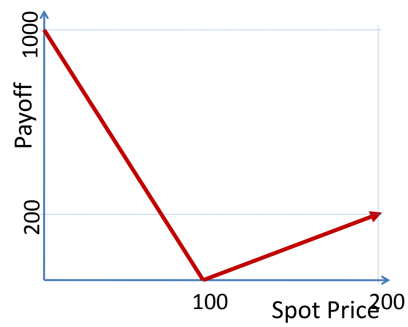
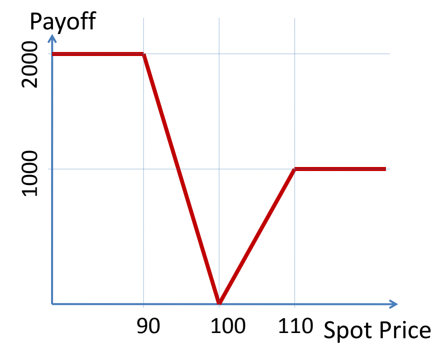
ACT 460 / STA 2502 Stochastic Methods for Actuarial Science
Problem Set #1 – due Tuesday, Oct 6 at 2PM:
ACT460 - Hand in only questions marked with (**).
STA2502 - IN ADDITION, hand in questions marked with (!!).

1. Construct each of the following payoffs using only stock, bonds, calls, and puts:







2. Use the Excel file `Porfolio.xls` to plot the price versus spot level for each option in Q1 using the following sets of parameters (put each parameter set on a single plot):

   (a) $T = \{\frac{1}{4}, \frac{1}{2}, 1\}$; $\sigma = 20\%$; $r = 5\%$; $\delta = 3\%$

   (b) $T = 1$; $\sigma = \{10\%, 20\%, 30\%\}$; $r = 5\%$; $\delta = 3\%$

   (c) $T = 1$; $\sigma = 20\%$; $r = \{0\%, 5\%, 10\%\}$; $\delta = 3\%$

   (d) $T = 1$; $\sigma = 20\%$; $r = 5\%$; $\delta = \{0\%, 3\%, 6\%\}$

   [Note: δ is a dividend yield.]

3. Using a CRR tree, with $S = 100$, $\sigma = 50\%$, $r = 5\%$ and $\Delta t = \frac{1}{12}$, determine the value and replicating strategy for each of the following 3-month European options:

   (a) digital call struck at 100 (A digital call pays 1 if $S_T > K$, otherwise it pays nothing)

   (b) digital put struck at 100 (A digital put pays 1 if $S_T < K$, otherwise it pays nothing)

   (c) put struck at 100

   (d) call struck at 100

   (e) straddle struck at 100

   (f) strangle with $K_1 = 95$, $K_2 = 115$

   (g) bull spread with $K_1 = 95$, $K_2 = 115$

   Suppose that the market prices for all of the above options are 10% higher than the no arbitrage prices, construct arbitrage strategies for each option.

4. Suppose a market has two risky assets A_t and B_t and the money-market account with risk-free rate of zero. Furthermore assume that $A_{t_n} = A_{t_{n-1}} e^{x_n \sigma_A \sqrt{\Delta t}}$ and $B_{t_n} = B_{t_{n-1}} e^{y_n \sigma_B \sqrt{\Delta t}}$ where $t_k = k\Delta t$ and $(x_1, y_1), \ldots, (x_N, y_N)$ are i.i.d joint Bernoulli r.v. with risk-neutral probabilities

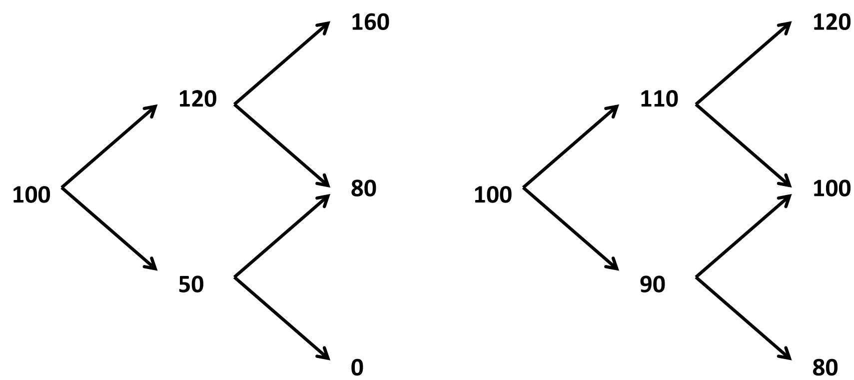
$$\mathbb{Q}(x_1 = +1, y_1 = +1) = q_1$$
$$\mathbb{Q}(x_1 = +1, y_1 = -1) = q_2$$
$$\mathbb{Q}(x_1 = -1, y_1 = +1) = q_3$$
$$\mathbb{Q}(x_1 = -1, y_1 = -1) = q_4$$

Find an expression for the no arbitrage bounds on the *instantaneous correlation* between asset A and asset B to lowest order in Δt, i.e. find the no arbitrage bounds on

$$\rho = \frac{Cov[\ln(A_{t_k}/A_{t_{k-1}})\ ;\ \ln(B_{t_k}/B_{t_{k-1}})]}{\sqrt{Var[\ln(A_{t_k}/A_{t_{k-1}})]\ Var[\ln(B_{t_k}/B_{t_{k-1}})]}} \ .$$

What are the bounds for the case $\sigma_A = 20\%$, $\sigma_B = 15\%$, and $\Delta t = 1/252$? Any comments?

5. The following two assets are being actively traded in a two-period binomial market economy. Asset A behaves like a stock which may default, while asset B behaves "normally".



(a) Asset A (b) Asset B

(a) Determine the probabilities induced by using asset B as a numeraire asset.

(b) **[5]**** Determine the risk-neutral probabilities and the implied risk-free rate over each branch of the model.
[Note: that the risk-free rate may differ from branch to branch – but at each node the risk-free rate used for discounting must be the same.]

(c) **[5]**** Compute the price and the replication strategy for a two-period European put option on asset A struck at 90.
[NOTE: The replication strategy must be specified at all nodes in the tree.]

(d) **[5]**** Compute the price and the replication strategy for a two-period American put option on asset A struck at 90. *[NOTE: The replication strategy must be specified at all nodes in the tree. Be careful at nodes where the option is exercised.]*

6. Assume that a stock price follows the continuous limit of the CRR tree and the continuous risk-free rate is r. Determine the value (at $t = 0$) of a contingent claim having the following payoff at time T:

   (a) S_T^α

   (b) $\mathbb{I}(S_T > K)$

   (c) **[5]** ** $S_T \, \mathbb{I}(S_T > K)$

   (d) **[5]** ** $S_T \, \mathbb{I}(S_T > S_U)$ where $U < T$.

7. **[5]** **!!** Let $\{t_j : j = 0, \ldots, m\}$ be an ordered series of times $t_0 = 0 < t_1 < t_2 < \cdots < t_m = T$. Suppose that an asset's price is modeled as the continuous time limit of the CRR model. Then define $\overline{S}(n)$ as the geometric average of the asset's price over the first n ordered times ($n \leq m$). That is, $\overline{S}(n) := \left(\prod_{j=1}^{n} S(t_j)\right)^{1/n}$. Determine the value of a call option written on $\overline{S}(n)$ with strike K maturing at T.
   [Hint: What is the distribution of $\overline{S}(n)$?]