Tuesday, December 07, 2010
2:09 PM

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$$

$$Y_t = F(t, X_t) \,, \qquad \underset{\in C^{1,2}}{F(t,x)} : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$$

Ito's lemma says:

$$dY_t = \partial_t F\, dt + \partial_x F\, dX_t + \underbrace{\tfrac{1}{2}\sigma^2(t, X_t)\, \partial_{xx} F\, dt}_{\text{Ito Correction.}}$$

$$dX_t = \mu^X(t, X_t, Y_t)\, dt + \sigma^X(t, X_t, Y_t)\, dW_t^X$$

$$dY_t = \mu^Y(t, X_t, Y_t)\, dt + \sigma^Y(t, X_t, Y_t)\, dW_t^Y$$

ρ

$$g_t = F(t, X_t, Y_t) \,, \qquad \underset{\in C^{1,2,2}}{F(t,x,y)} : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$$

2-D Ito's lemma:

$$dg_t = \partial_t F\, dt + \partial_x F\, dX_t + \partial_y F\, dY_t$$

$$\left.\begin{aligned} &+ \tfrac{1}{2}(\sigma^X)^2\, \partial_{xx} F\, dt + \tfrac{1}{2}(\sigma^Y)^2\, \partial_{yy} F\, dt \\ &+ \rho\, \sigma^X \sigma^Y\, \partial_{xy} F\, dt \end{aligned}\right\} \text{Ito correction terms}$$

$$\frac{dA_t}{A_t} = \mu_A\, dt + \sigma_A\, dW_t^A$$

ρ

$$\frac{dB_t}{B_t} = \mu_B\, dt + \sigma_B\, dW_t^B$$

$$g_t = A_t B_t = F(t, A_t, B_t) \text{ with } F(t,a,b) = ab$$

$$\Rightarrow dg_t = \underset{(\partial_t f\, dt)}{0\, dt} + \underset{(\partial_a f\, dA_t)}{B_t\, dA_t} + \underset{(\partial_b f\, dB_t)}{A_t\, dB_t}$$

$$+ \frac{1}{2}\sigma_a^2 A_t^2 \underset{\partial_{aa} f}{(0)}\, dt + \frac{1}{2}\sigma_b^2 B_t^2 \underset{\partial_{bb} f}{(0)}\, dt$$

$$+ \sigma_a A_t \sigma_b B_t \rho \underset{\partial_{ab} f}{(1)}\, dt$$

$$= B_t(\mu_a A_t\, dt + \sigma_a A_t\, dW_t^A)$$

$$+ A_t(\mu_b B_t\, dt + \sigma_b B_t\, dW_t^B)$$

$$+ \sigma_a \sigma_b \rho A_t B_t\, dt$$

$$\Rightarrow \boxed{\frac{dg_t}{g_t} = (\mu_a + \mu_b + \sigma_a \sigma_b \rho)\, dt + \sigma_a\, dW_t^A + \sigma_b\, dW_t^B}$$

recall : $\dfrac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$

let's define $X_t = \ln S_t$

[$\dfrac{dS_t}{S_t}$ looks like $d \ln S_t$]

$$dX_t = 0\, dt + \frac{1}{S_t} \cdot dS_t + \overbrace{\frac{1}{2}\sigma^2 S_t^2 \left(-\frac{1}{S_t^2}\right)}^{-\frac{1}{2}\sigma^2} dt$$

$$= (\mu - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, dW_t$$

$$\Rightarrow X_t - X_0 = (\mu - \tfrac{1}{2}\sigma^2)\, t + \sigma W_t$$

$$\Rightarrow S_t = S_0\, e^{(\mu - \frac{1}{2}\sigma^2) t + \sigma W_t}$$

---

let $F_t = \ln g_t$

$$dF_t = 0\,dt + \frac{1}{g_t}\,dg_t + \frac{1}{2}\underbrace{\left(\sigma_a^2 + \sigma_b^2 + 2\sigma_a\sigma_b\rho\right)g_t^2}\left(-\frac{1}{g_t^2}\right)dt$$

$$\sigma_a g_t\, dW_t^A + \sigma_b g_t\, dW_t^B$$

$$= (\mu_a + \mu_b - \tfrac{1}{2}\sigma_a^2 - \tfrac{1}{2}\sigma_b^2)\,dt$$

$$+ \sigma_a\, dW_t^A + \sigma_b\, dW_t^B$$

$$\Rightarrow \quad F_t - F_0 = (\mu_a + \mu_b - \tfrac{1}{2}\sigma_a^2 - \tfrac{1}{2}\sigma_b^2)\,t + \sigma_a W_t^A + \sigma_b W_t^B$$

$$\Rightarrow \quad g_t = \cancel{g_0}\, \underbrace{e^{(\mu_a - \frac{1}{2}\sigma_a^2)t + \sigma_a W_t^A}}_{\frac{A_t}{\cancel{A_0}}} \; \underbrace{e^{(\mu_b - \frac{1}{2}\sigma_b^2)t + \sigma_b W_t^B}}_{\frac{B_t}{\cancel{B_0}}}$$

$g_t = A_t / B_t$ what is the SDE for g_t?

$$= f(t, A_t, B_t) \quad \text{with} \quad f(t, a, b) = a/b$$

$$dg_t = 0\,dt + \frac{1}{B_t}\cdot dA_t - \frac{A_t}{B_t^2}\,dB_t$$

$$+ \tfrac{1}{2}(0)\,\sigma_a^2 A_t^2\,dt + \tfrac{1}{2}\left(\frac{2A_t}{B_t^3}\right)\cdot \sigma_b^2 B_t^2\,dt$$

$$+ \left(-\frac{1}{B_t^2}\right)\sigma_a A_t\, \sigma_b B_t\, \rho\, dt$$

$$= \frac{A_t}{B_t}\left(\mu_a\,dt + \sigma_a\,dW_t^A\right)$$

$$- \frac{A_t}{B_t}\left(\mu_b\,dt + \sigma_b\,dW_t^B\right)$$

$$+ \frac{A_t}{B_t}\,\sigma_b^2\,dt - \sigma_a\sigma_b\rho\,\frac{A_t}{B_t}\,dt$$

$$\Rightarrow \boxed{\frac{dg_t}{g_t} = (\mu_a - \mu_b + \sigma_b^2 - \sigma_a \sigma_b \rho)\, dt + \sigma_a dW_t^A - \sigma_b dW_t^B}$$

↳ try to solve and find that indeed

$$g_t = A_t / B_t$$

this is a GBM driven 2 B. motions

$$\Rightarrow g_t = g_0 \exp\Big\{\big[(\mu_a - \mu_b + \sigma_b^2 - \cancel{\sigma_a\sigma_b}\rho) - \tfrac{1}{2}(\sigma_a^2 + \sigma_b^2 - \cancel{2\sigma_a\sigma_b}\rho)\big]t + \sigma_a W_t^A - \sigma_b W_t^B\Big\}$$

$$= g_0 \exp\Big\{\underbrace{\big[(\mu_a - \tfrac{1}{2}\sigma_a^2) + (\mu_b - \tfrac{1}{2}\sigma_b^2)\big]}_{\bar{\mu}} t + \sigma_a W_t^A - \sigma_b W_t^B\Big\}$$

$$\ln(g_t/g_0) \sim N\big(\bar{\mu}\, t \,;\, (\sigma_a^2 + \sigma_b^2 - 2\sigma_a\sigma_b\rho)\, t\big)$$

Tuesday, December 07, 2010
3:04 PM

$$S_t = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

$$\stackrel{d}{=} e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\, Z}$$

$$Z \underset{P}{\sim} N(0,1)$$

$$\begin{array}{ccc} & & \uparrow \mathbb{Q} \\ \vdash\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!+ & \!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!+ & \!\!\!-\!\!\!-\!\!\!- \\ 0 & S & T \end{array}$$

(b) *[5]* Determine the price at time $t = 0$ of an option which pays

$$\varphi = \frac{V_S}{U_T}\,\mathbb{I}(V_S > \gamma)$$

at the maturity date T and $T > S > 0$. Here, γ is a positive constant.

$$\frac{dU_t}{U_t} = \cancel{\alpha}^{\,\hat{r} = 0}\, dt + \sigma\, d\widehat{X}_t\,, \qquad \frac{dV_t}{V_t} = \cancel{\beta}^{\,\hat{r} = 0}\, dt + \eta\, d\widehat{Y}_t\,,$$

The risk-free rate is zero.

$$\hookrightarrow \quad U_T = U_S\, e^{-\frac{1}{2}\sigma^2 (T-S) + \sigma(\hat{X}_T - \hat{X}_S)} \overset{d}{=} U_S\, e^{-\frac{1}{2}\sigma^2 (T-S) + \sigma\sqrt{T-S}\, Z}$$

$$Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$P_0 = \mathbb{E}^{\mathbb{Q}}\left[\frac{V_S}{U_T} \mathbb{1}_{V_S > \gamma} \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\frac{V_S}{U_T} \mathbb{1}_{V_S > \gamma} \,\middle|\, V_S, U_S \right]\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[V_S \mathbb{1}_{V_S > \gamma}\, \underline{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{U_T} \,\middle|\, U_S, V_S \right]} \right]$$

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{U_T} \,\middle|\, U_S \right] \overset{?}{=} \frac{1}{\mathbb{E}^{\mathbb{Q}}[U_T | U_S]} = \frac{1}{U_S} \quad \times$$

$$\mathbb{E}^{\mathbb{Q}}\left[U_T \,|\, U_S \right] = U_S \quad \checkmark$$

so

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{U_T} \middle| U_S \right] = U_S^{-1} e^{\frac{1}{2}\sigma^2 (T-S)}\, \mathbb{E}^{\mathbb{Q}}\left[e^{-\sigma\sqrt{T-S}\, Z} \right]$$

$$= U_S^{-1} e^{\frac{1}{2}\sigma^2 (T-S)}\, e^{\frac{1}{2}\sigma^2 (T-S)}$$

$$= U_S^{-1} e^{\sigma^2 (T-S)}$$

$$P_0 = \mathbb{E}^{\mathbb{Q}}\left[\frac{V_S}{U_S} \mathbb{1}_{V_S > \gamma}\, e^{\sigma^2 (T-S)} \right]$$

$$V_S \overset{d}{=} V_0\, e^{-\frac{1}{2}\eta^2 S + \eta\sqrt{S}\, Z_1} \qquad Z$$

$$U_s \overset{d}{=} U_0 \, e^{-\frac{1}{2}\sigma^2 s + \sigma\sqrt{s}\, Z_2}$$

$Z_2 \to \underline{\rho Z + \sqrt{1-\rho^2}\, Z^\perp}$ $\quad Z, Z^\perp \sim \mathcal{N}(0,1)$ $\mathbb{Q}$ iid.

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \underset{\mathbb{Q}}{\sim} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} \rho & 1 \\ 1 & \rho \end{pmatrix} \right)$$

$$\frac{V_s}{U_s} \mathbb{1}_{V_s > \gamma} \overset{d}{=} \frac{V_0}{U_0} \cdot e^{-\frac{1}{2}(\eta^2 - \sigma^2) s} \cdot e^{\eta\sqrt{s}\, Z}$$

$$\times \; e^{-\sigma\sqrt{s}\,(\rho Z + \sqrt{1-\rho^2}\, Z^\perp)}$$

$$\mathbb{1}\left(V_0 \, e^{-\frac{1}{2}\eta^2 s + \eta\sqrt{s}\, Z} > \gamma \right) \to Z > z_*$$

$$z_* = \frac{\ln(\gamma/V_0) + \frac{1}{2}\eta^2 s}{\eta\sqrt{s}}$$

$$= \frac{V_0}{U_0} e^{-\frac{1}{2}(\eta^2 - \sigma^2) s} \cdot \underbrace{e^{-\sigma\sqrt{s}\, Z^\perp \sqrt{1-\rho^2}}}$$

$$\cdot \underbrace{e^{(\eta - \sigma\rho)\sqrt{s}\, Z} \, \mathbb{1}_{Z > z_*}}$$

independent

$$\Rightarrow P_0 = \frac{V_0}{U_0} e^{-\frac{1}{2}(\eta^2 - \sigma^2) s} \, \underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\sigma\sqrt{(1-\rho^2) s}\, Z^\perp} \right]}_{A}$$

$$\times \underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{(\eta - \sigma\rho)\sqrt{s}\, Z} \, \mathbb{1}_{Z > z_*} \right]}_{B}$$

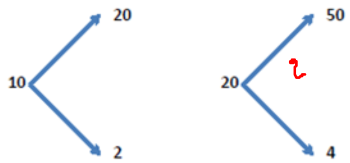
$$A = e^{\frac{1}{2}\sigma^2 (1-\rho^2) s}$$

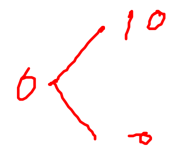
$$B = \int_{z_*}^{\infty} e^{(\eta - \sigma\rho)\sqrt{s}\, z} \, \frac{e^{-\frac{1}{2} z^2}}{\sqrt{2\pi}} \, dz$$

$$= \int_{z_*}^{\infty} e^{-\frac{1}{2}(z - (\eta - \sigma\rho)\sqrt{s})^2 + (\eta - \sigma\rho)^2 s} \, \frac{dz}{\sqrt{2\pi}}$$

$$= e^{(\eta - \sigma\rho)^2 s} \, \Phi\left(-z_* + (\eta - \sigma\rho)\sqrt{s} \right)$$

Tuesday, December 07, 2010
4:03 PM





×−2

×1

$$\begin{cases} 20q + 2(1-q) = 10(1+r) \\ 50q + 4(1-q) = 20(1+r) \end{cases}$$

$$\frac{4+46q}{2+18q} = 2$$

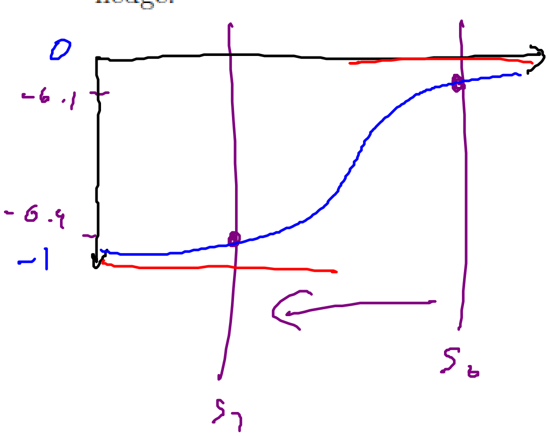
$$\Rightarrow \quad 4 + 46q = 4 + 36q$$

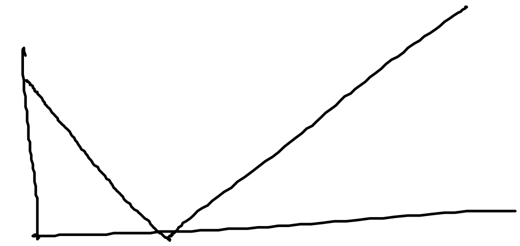
$$\Rightarrow \quad 46 = 36$$

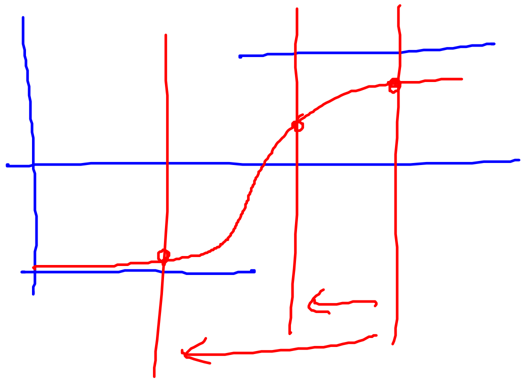
unless $q=0 \Rightarrow q \notin (0,1)$ so $\exists$ an arb!

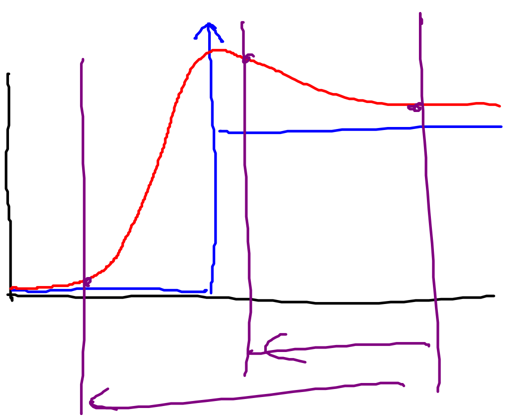
(b) [T] [F]

You have sold a put option on XYZ shares and you are simultaneously delta-hedging the position. Suppose that important (unexpected) news arrives declaring poor sales of XYZ products resulting in a drop in share value. You must sell shares of XYZ to maintain your hedge.









(c) [T] [F]

If $X_t = \mu t + W_t$ where $\mu > 0$ and W_t is a standard Brownian motion, then the variance of X_t is equal to t.

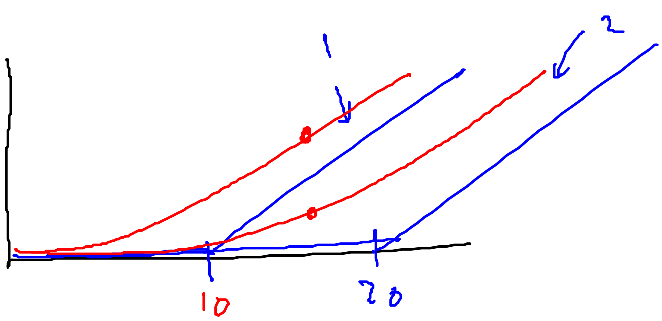
$$\mathbb{V}[\mu t + W_t] = \mathbb{V}[W_t] = t$$

(d) [T] [F]

Delta hedging using a move-based approach always outperforms hedging using a time-based approach.

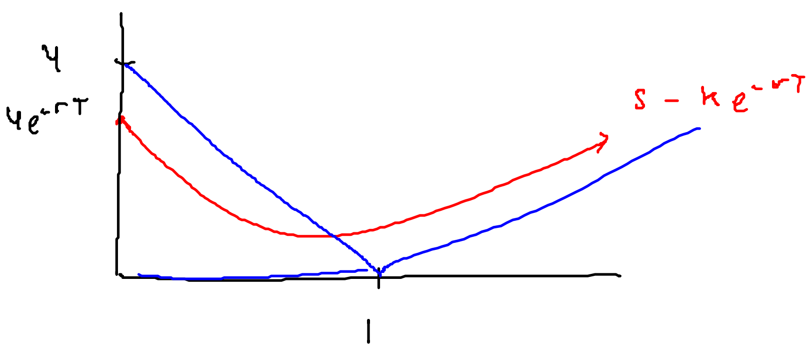
(e) [T] [F]

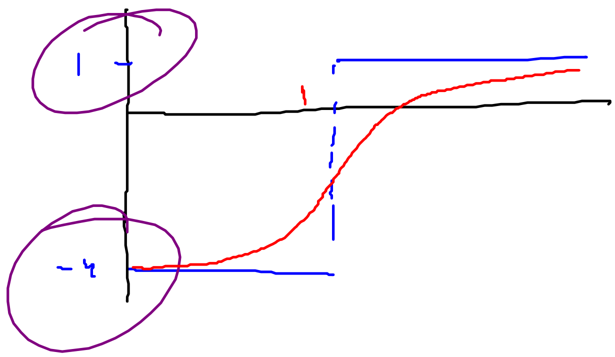
Suppose that a call option struck at 10 is selling for 1; while a call option struck at 20 is selling for 2. Both call options have the same maturity. This economy admits an arbitrage.



$C_{10} > C_{20}$

(a) *[5]* Consider the following portfolio: 4 long puts struck at 1 and one long call struck at 1. Sketch the **delta** of the portfolio (i) at maturity (ii) 1-year from maturity on the same graph below. Label any important points clearly.





[5] Sketch the **gamma** of an **asset-or-nothing call option** struck at 1 (i) at maturity (ii) 1-year from maturity on the same graph below. Label any important points clearly. *[Recall that an asset-or-nothing call has a payoff of S_T if $S_T > K$, otherwise it pays 0.]*