Tuesday, September 28, 2010
2:09 PM

Value a call on an asset S, strike K, mat T.

$$V_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)_+ \right]$$

$$S_T \overset{d}{=} S_0\, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, z}, \quad z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)\,\mathbb{1}_{S_T > K} \right]$$

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0, & \text{otherwise} \end{cases}$$

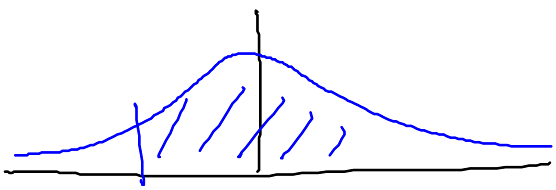
$$= \mathbb{E}^{\mathbb{Q}}\left[S_T\,\mathbb{1}_{S_T>K} \right] - K\,\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{S_T>K} \right]$$

$\hookrightarrow$ $\mathbb{Q}(S_T > K)$

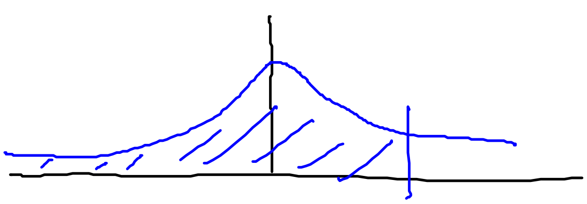
$$= \mathbb{Q}\Bigg(z > \underbrace{- \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{z_*} \Bigg)$$

$$= \Phi\left(\frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

$\hookrightarrow$ std. normal cdf



z_*



$-z_*$

$$\mathbb{E}^{\mathbb{Q}}\left[S_T\,\mathbb{1}_{S_T>K} \right] = \int_{-\infty}^{\infty} S_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, z}\, \mathbb{1}_{z>z^*}\, \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\, dz$$

$$= S_0\, e^{(r-\frac{1}{2}\sigma^2)T} \underbrace{\int_{z^*}^{\infty} e^{\sigma\sqrt{T} z - \frac{1}{2}z^2}\, \frac{dz}{\sqrt{2\pi}}}_{I}$$

$$I = \int_{z^*}^{\infty} e^{-\frac{1}{2}(\underbrace{z - \sigma\sqrt{T}}_{z'})^2 + \frac{1}{2}\sigma^2 T}\, \frac{dz}{\sqrt{2\pi}}$$

$$= e^{\frac{1}{2}\sigma^2 T} \int_{z^* - \sigma\sqrt{T}}^{\infty} e^{-\frac{1}{2}(z')^2} \frac{dz'}{\sqrt{2\pi}}$$

$$= e^{\frac{1}{2}\sigma^2 T} \, \Phi\left(\sigma\sqrt{T} - z^* \right)$$

$$= e^{\frac{1}{2}\sigma^2 T} \, \Phi\left(\frac{\ln S/K + (r + \frac{1}{2}\sigma^2) T}{\sigma\sqrt{T}} \right)$$

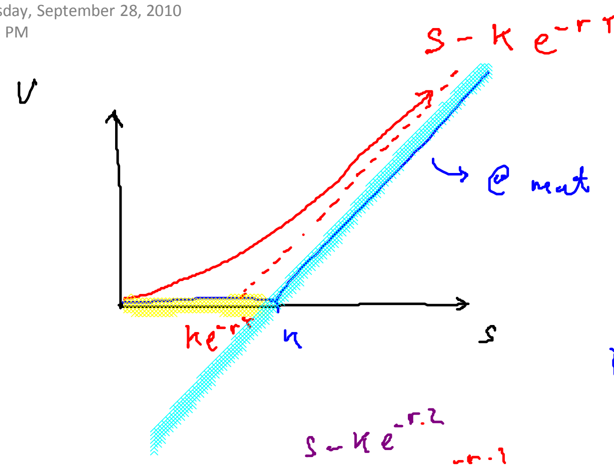
$$\boxed{\begin{aligned} V_0^c &= S_0 \, \Phi(d_+) - K e^{-rT} \, \Phi(d_-) \\ d_\pm &= \frac{\ln(S_0/K) + (r \pm \frac{1}{2}\sigma^2) T}{\sigma\sqrt{T}} \end{aligned}}$$

Black-Scholes Formula for a call option.

put "

$$V_0^p = K e^{-rT} \, \Phi(-d_-) - S_0 \, \Phi(-d_+)$$

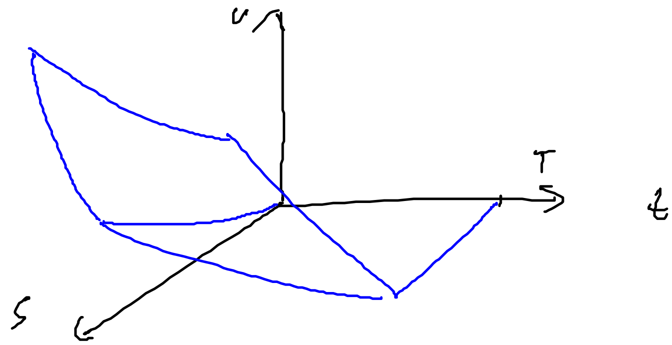
Tuesday, September 28, 2010
2:43 PM

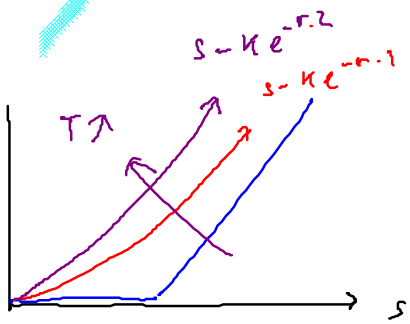


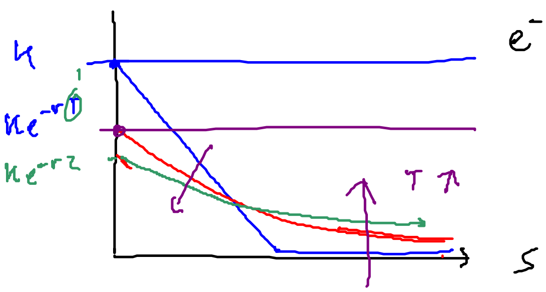
$$\Phi^c \ge S_T - K$$

$$\downarrow \qquad \downarrow$$

$$V^c \ge S_0 - Ke^{-rT}$$





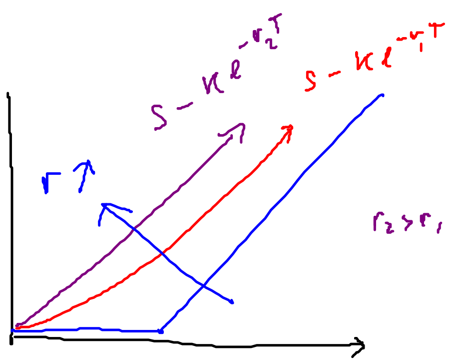


$$e^{-rT}\ \mathbb{E}^{\mathbb{Q}}\left[\left(K - S_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z}\right)_+\right]$$

$$\Phi^P \le K$$

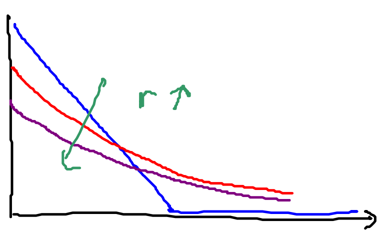
$$\downarrow \qquad \downarrow$$

$$V^P \le Ke^{-rT}$$

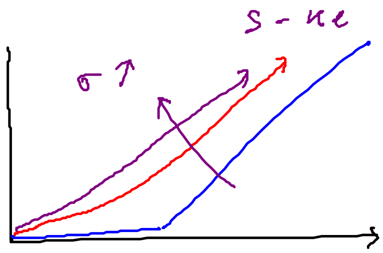


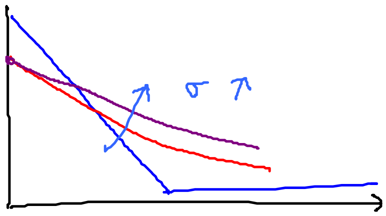
$$V_c = e^{-rT}\ \mathbb{E}^{\mathbb{Q}}\left[\left(S_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z} - K\right)_+\right]$$

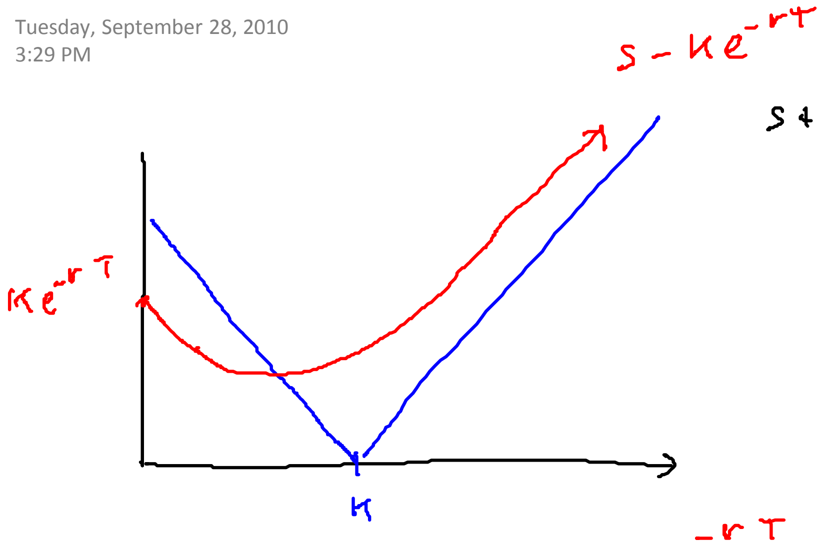
$$= \mathbb{E}^{\mathbb{Q}}\left[\left(S_0\, e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}Z} - Ke^{-rT}\right)_+\right]$$



$\cdot \ {}^{-rT}$

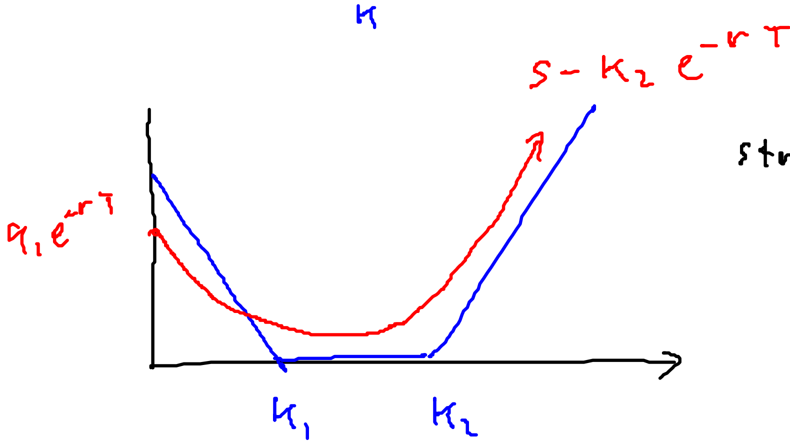

S - Ke
σ ↗


σ ↗

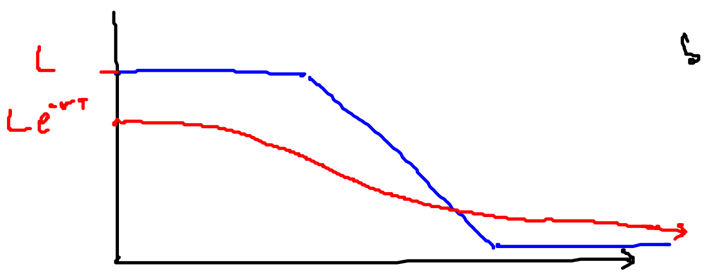


Straddle

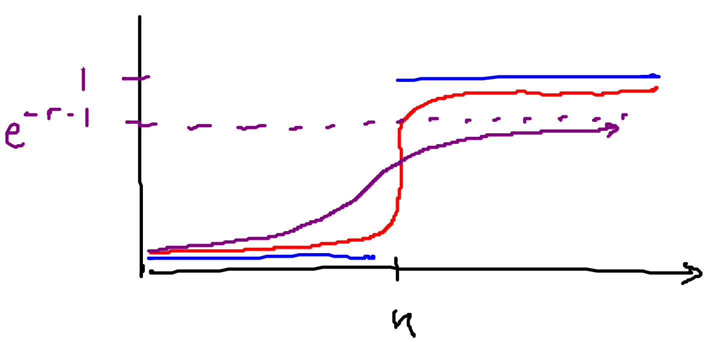
put K
call K

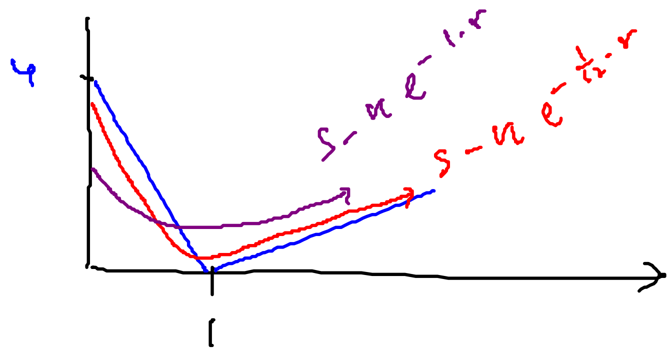


Strangle



bear spread

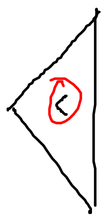


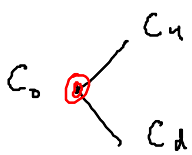

4
s - n e^{-1·r}
s - n e^{-1/√2·r}
1

Tuesday, September 28, 2010
3:44 PM

American styled claims: can exercise at any time.

in principle can have black-out periods.



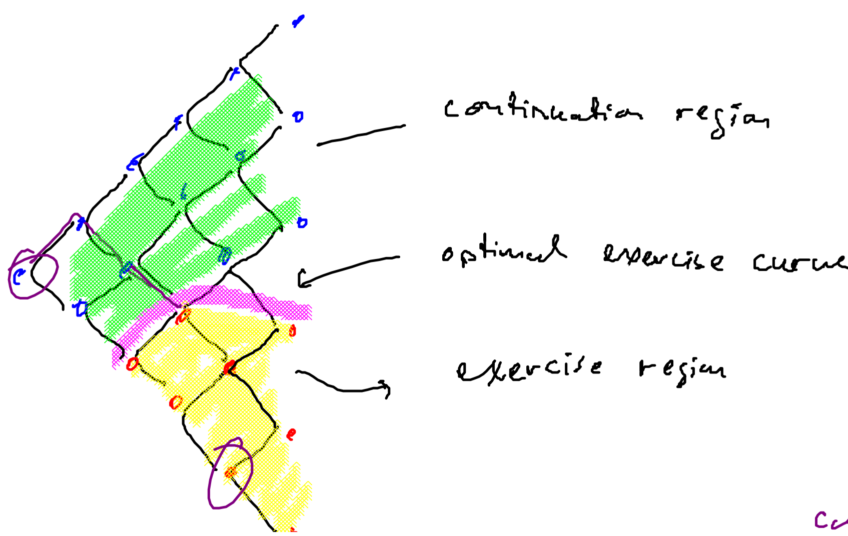


① hold H_0 or ② exercise E_0

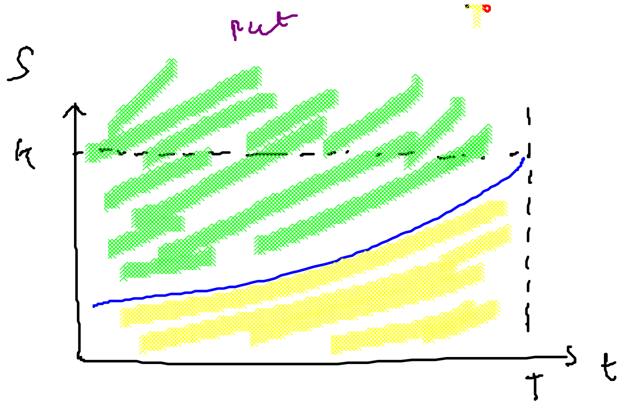
$$C_0 = \max(H_0, E_0)$$

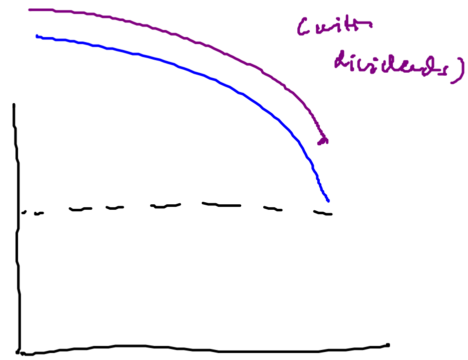
(Put) $E_0 = (K - S_0)_+$

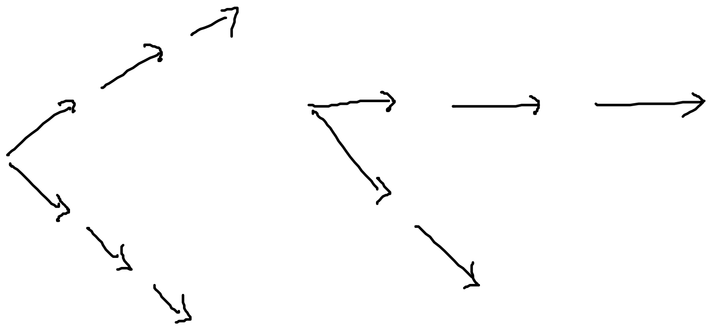
$$H_0 = e^{-r\Delta t}\left(q\, C_u + (1-q)\, C_d\right)$$

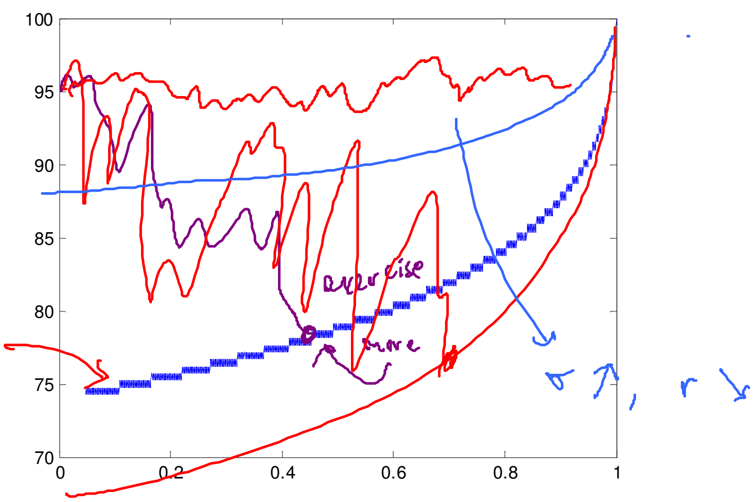


call



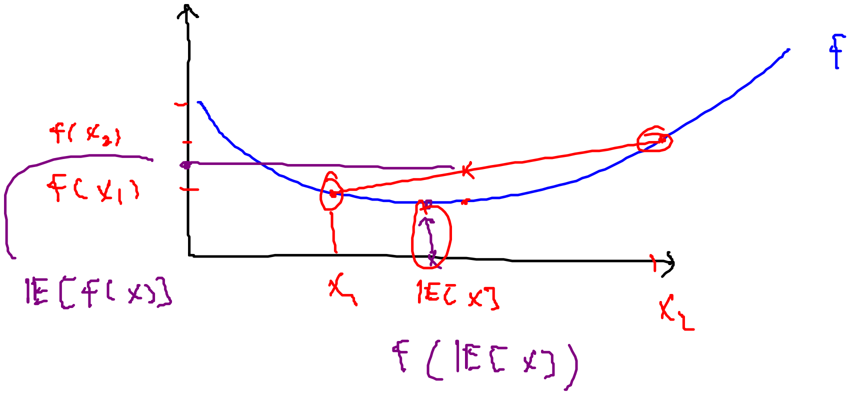






$F(\cdot): \mathbb{R} \mapsto \mathbb{R}$ convex

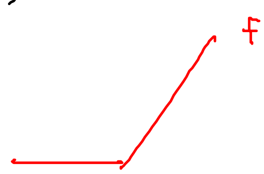
$$\boxed{\mathbb{E}[F(X)] \geq F(\mathbb{E}[X])}$$



Q: show that it is never optimal to exercise a call option early.
[no dividends]

① $\mathbb{E}^{\tilde{Q}}[S_T] = S_0 e^{rT}$

② $\mathbb{E}[f(x)] \geq f(\mathbb{E}[x])$



$$H_0 = e^{-rT}\mathbb{E}^{\tilde{Q}}[(S_T - K)_+] \qquad f(x) = (x-K)_+ \text{ is convex.}$$

$$\therefore \quad \mathbb{E}^{\tilde{Q}}[f(S_T)] \geq f(\mathbb{E}^{\tilde{Q}}[S_T]) = f(S_0 e^{rT})$$

Jensen's inequality

$$= (S_0 e^{rT} - K)_+$$

$$\Rightarrow \quad H_0 \geq e^{-rT}(S_0 e^{rT} - K)_+ = (S_0 - Ke^{-rT})_+$$

$$\geq (S_0 - K)_+ = E_0$$

$\therefore$ never optimal to exercise.