Q2

Thursday, November 19, 2009
1:19 PM

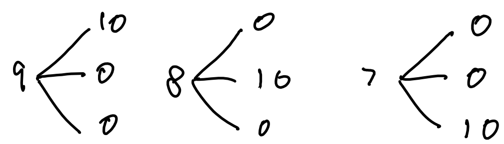
2. *[10]* Please indicate true or false (**no explanations required**).

   +2 for correct answer; −0.5 for incorrect answer; 0 for no answer.

   (a) [T] [F]

   In an economy with three tradable assets, it is never possible to replicate contingent claims written on one of the assets.

false. e.g.

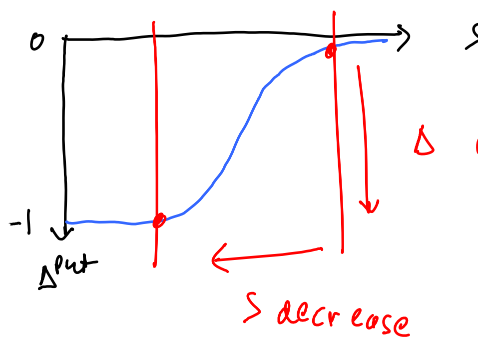


can replicate anything.

(b) [T] [F]

You have sold a put option on XYZ shares and you are simultaneously delta-hedging the position. Suppose that important (unexpected) news arrives declaring poor sales of XYZ products resulting in a drop in share value. You must sell shares of XYZ to maintain your hedge.

sold a put ⇒ replicating a put to hedge

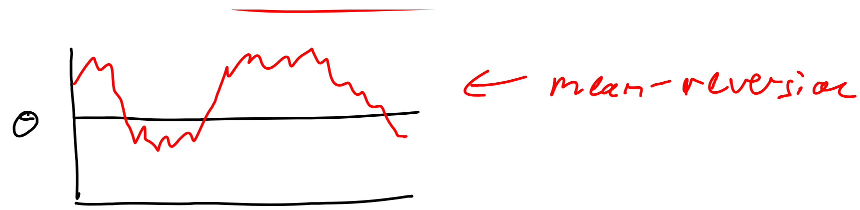


Δ decreases ⇒ sell shares to maintain hedge.

(c) [T] [F]

If interest rates are modeled as $dr_t = \theta\,dt + \sigma\,dW_t$ where W_t is a Brownian motion, then interest rates mean-revert.

false since $r_t = \theta t + \sigma W_t$ it is a Brownian motion and is never pulled back to the level θ.



(d) [T] [F]

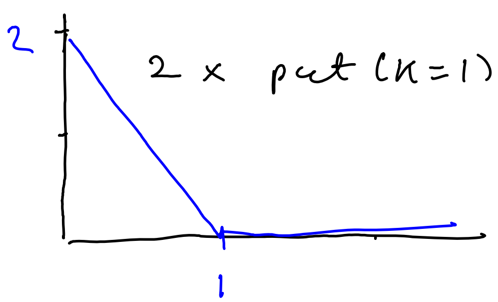
If the real-world evolution of share prices evolves with a vol of 20% and you delta-gamma hedge a put option with a vol of 25% on a daily basis, then the net PnL will be symmetric.

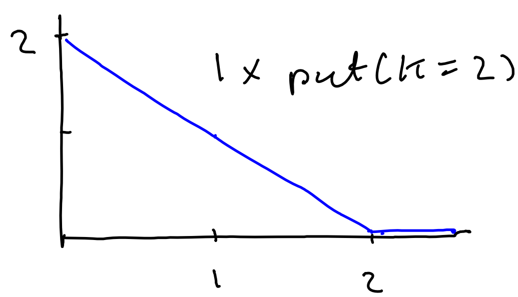
False. but you have not seen this in our lectures this year!

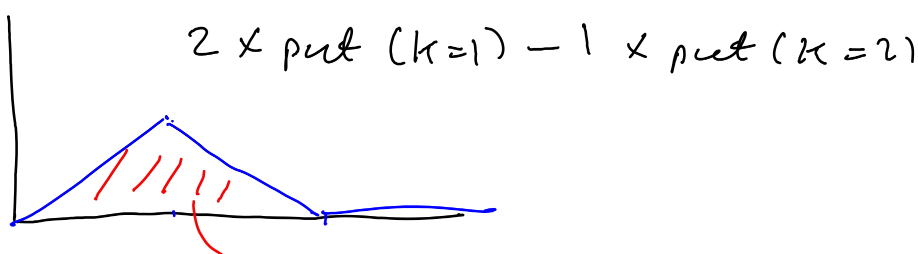
(e) [T] [F]

Suppose that a put option struck at 1 is selling for 0.10; while a put option struck at 2 is selling for 0.2. Both puts have the same maturity. This economy admits an arbitrage.

consider







$\mathbb{P}(V_1 > 0) > 0$

$\mathbb{P}(V_1 \geq 0) = 1$

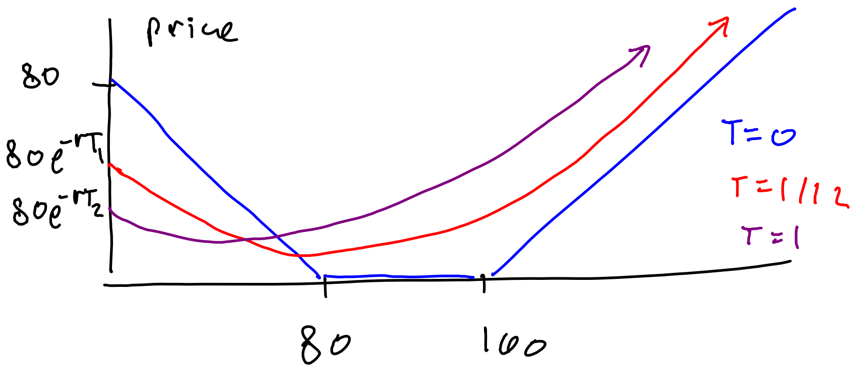
$V_0 = 2 \times 0.1 - 0.2 = 0$

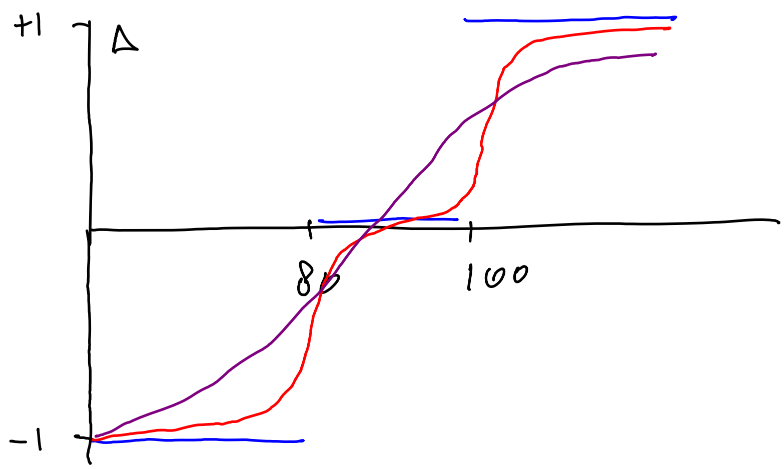
$\therefore$ arb.

Q3

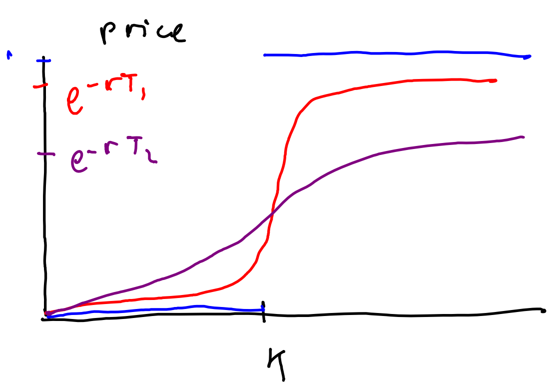
Thursday, November 19, 2009
1:31 PM

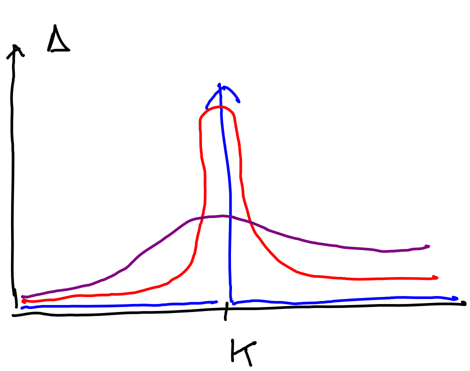
3. (a) *[5]* Consider the following portfolio: long put struck at 80 and a long call struck at 100. Sketch the **delta** of the portfolio (i) at maturity (ii) 1-month from maturity (iii) 1-year from maturity all on the same graph. Label any important points clearly.

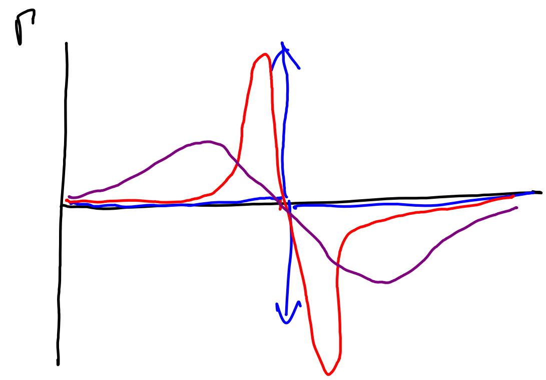




(b) *[5]* Sketch the **gamma** of a **digital call** option (i) at maturity (ii) 1-month from maturity (iii) 1-year from maturity all on the same graph. Label any important points clearly. *[Recall that a digital call option pays 1 at maturity if the asset price exceeds the strike K otherwise it pays nothing.]*





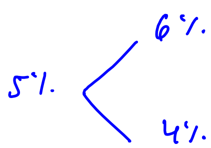

r

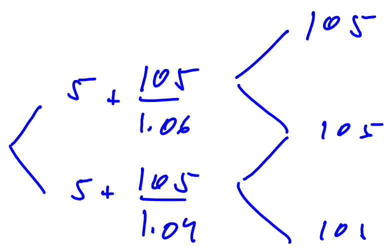
Q4

Thursday, November 19, 2009
1:40 PM

4. Consider a simple two-step binomial model of interest rates in which $r_0 = 5\%$, and $r_n = r_{n-1} \pm 1\%$. (Treat these rates as per period discount rates – e.g. discounting over the first period is $1/1.05$).

   (a) *[5]* Determine the risk-neutral branching probabilities consistent with a market price of 100 for a coupon bearing bond which pays 5 at $t = 1$ and 105 at $t = 2$.





$$100 = \frac{5}{1.05} + \frac{105}{1.05}\left[\frac{1}{1.06} q + \frac{1}{1.04}(1-q)\right]$$

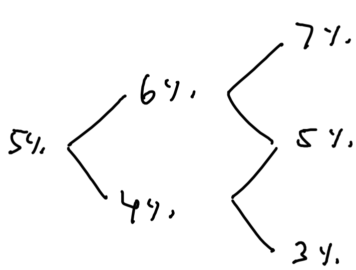
$$\Rightarrow \quad \frac{100}{105} = \left(\frac{1}{1.06} - \frac{1}{1.04}\right) q + \frac{1}{1.04}$$

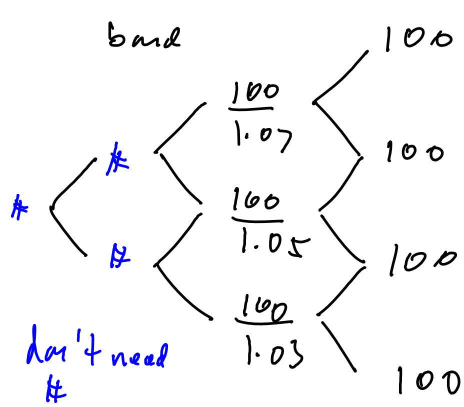
$$\Rightarrow \quad q = \left(\frac{100}{105} - \frac{1}{1.04}\right) \Big/ \left(\frac{1}{1.06} - \frac{1}{1.04}\right)$$

$$= 0.5048$$

(b) *[5]* Suppose that the risk-neutral branching probabilities are $q = 1/2$.

Consider a European call option on a 3-period bond with notional 100. The option matures at $t = 2$ and the strike of the option is 95. Determine the value of the option.





call

$$
\begin{tikzcd}[column sep=small]
 & & \left(\frac{100}{1.07}-95\right)_+ = 0 \\
 & C_u \arrow[ur, no head] \arrow[dr, no head] & \\
C \arrow[ur, no head] \arrow[dr, no head] & & \left(\frac{100}{1.05}-95\right)_+ = 0.238 \\
 & C_d \arrow[ur, no head] \arrow[dr, no head] & \\
 & & \left(\frac{100}{1.03}-95\right)_+ = 2.087
\end{tikzcd}
$$

$$C_u = \frac{1}{1.06} \cdot \left[\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0.238 \right]$$

$$= 0.045$$

$$C_d = \frac{1}{1.04} \left[\frac{1}{2} \cdot 0.238 + \frac{1}{2} \cdot 2.087 \right]$$

$$= 1.12$$

$$C = \frac{1}{1.05} \left[\frac{1}{2} \cdot 0.045 + \frac{1}{2} \cdot 1.12 \right]$$

$$= 0.55$$

Q5

Thursday, November 19, 2009
1:47 PM

5. Consider a digital call option in the Black-Scholes model with <u>zero interest rates</u>.

(a) [5] Show that the price of the digital call option is

$$V(S,t) = \Phi(d_-)\ , \quad d_- = \frac{\ln(S/K)}{\sigma(T-t)^{1/2}} - \tfrac{1}{2}\sigma(T-t)^{1/2}\ .$$

$V(S,t)$

$= \mathbb{E}^{\mathbb{Q}}[\, \mathbb{1}_{S_T > K} \mid S_t = S]$

$= \mathbb{Q}(S_T > K \mid S_t = S)$

$= \mathbb{Q}(S\, e^{-\frac{1}{2}\sigma^2 (T-t) + \sigma\sqrt{T-t}\, Z} > K)$

since $S_T \overset{d}{=} S_t\, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z}$, $Z \underset{\mathbb{Q}}{\sim} N(0,1)$

(and $r=0$)

$\Rightarrow V(S,t) = \mathbb{Q}\left(Z > -\frac{(\ln(S/K) - \frac{1}{2}\sigma^2(T-t))}{\sigma\sqrt{T-t}}\right)$

$= \Phi\left(\frac{\ln(S/k)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma^2\sqrt{T-t}\right)$

(b) [5] Confirm that the price satisfies the Black-Scholes partial differential equation.
[Hint: use the fact that $\Phi''(x) = -x\Phi'(x)$]

B-S eqn:

$\partial_t V + r S \partial_S V + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V = r V$, set $r=0$

$\partial_t V = \Phi'(d_+)\ \partial_t d_+$

$\partial_S V = \Phi'(d_+)\ \partial_S d_+$

$\partial_{SS} V = \Phi''(d_+)\ (\partial_S d_+)^2 + \Phi'(d_+)\ \partial_{SS} d_+$

$$= \Phi'(d_+)\left[-d_+ (\partial_S d_+)^2 + \partial_{SS} d_+\right]$$

$$\partial_t d_+ = +\frac{1}{2}\frac{\ln(S/K)}{\sigma (T-t)^{3/2}} + \frac{1}{4}\frac{\sigma}{(T-t)^{1/2}}$$

$$\partial_S d_+ = \frac{1}{S\sigma\sqrt{T-t}}$$

$$\partial_{SS} d_+ = -\frac{1}{S^2\sigma\sqrt{T-t}}$$

$$\Rightarrow \quad \partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V$$

$$= \Phi'(d_+)\Bigg\{ \frac{1}{2}\cancel{\frac{\ln S/K}{\sigma (T-t)^{3/2}}} + \frac{1}{4}\frac{\sigma}{(T-t)^{1/2}} + \frac{1}{2}\sigma^2 S^2\left(-\left(\cancel{\frac{\ln S/K}{\sigma\sqrt{T-t}}} - \frac{1}{2}\sigma\sqrt{T-t}\right)\frac{1}{S^2\sigma^2(T-t)} - \frac{1}{S^2\sigma\sqrt{T-t}}\right)\Bigg\}$$

$$= 0$$

Q6

Thursday, November 19, 2009
1:56 PM

6. You are given that W_t and B_t are correlated Brownian motions with correlation ρ.

(a) *[5]* Obtain an integration by parts formula for $\int_0^t e^{W_s}\,dB_s$.

$$g(W_t, Z_t, t) = e^{W_t} B_t$$

$$dg = \left[e^{W_t} B_t + e^{W_t} \rho \right] dt + e^{W_t} B_t \, dW_t + e^{W_t} t^2 \, dB_t$$

$$\Rightarrow \int_0^t e^{W_s} s^2 \, dB_s = e^{W_t} t^2 B_t - \int_0^t e^{W_s} (B_s + \rho) \, ds - \int_0^t e^{W_s} s^2 B_s \, dW_s$$

(b) *[5]* Determine the mean and variance of $X_t = \int_0^t W_s\,dB_s - \int_0^t B_s\,dW_s$.

$$\mathbb{E}[X_t] = 0$$

$$\mathbb{V}[X_t] = \mathbb{V}\left[\int_0^t W_s \, dB_s\right] + \mathbb{V}\left[\int_0^t B_s \, dW_s\right] - 2\,\mathbb{C}\left[\int_0^t W_s \, dB_s , \int_0^t B_s \, dW_s\right]$$

$$= 2\int_0^t s \, ds - 2\rho \int_0^t s \, ds$$

$$= (1-\rho)\, t^2$$

Q7

Thursday, November 19, 2009
1:58 PM

7. *[10]* Suppose that two stocks U_t and V_t satisfy the following SDEs:

$$\frac{dU_t}{U_t} = \alpha\,dt + \sigma\,dX_t\,, \qquad \frac{dV_t}{V_t} = \beta\,dt + \eta\,dY_t\,,$$

where X_t and Y_t are $\mathbb{P}$-Wiener processes with correlation $d[X,Y]_t = \rho\,dt$ and $\alpha, \beta, \sigma, \eta$ are all constants. The risk-free rate is zero.

Determine the price at time $t = 0$ of an option which pays

$$\varphi = U_T \times \mathbb{I}_{V_S > \gamma}$$

at the maturity date T and $T > S > 0$. Here, γ is a constant.

$\frac{dU_t}{U_t} = r\,dt + \sigma\,d\hat{X}_t$ and $\frac{dV_t}{V_t} = r\,dt + \eta\,d\hat{Y}_t$ but $r = 0$

$\hat{X}_t, \hat{Y}_t$ are $\mathbb{Q}$-Wiener processes.

$$c_t = \mathbb{E}_0^{\mathbb{Q}}\left[U_T \mathbb{1}_{V_S > \gamma} \right]$$

$$= \mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{E}_S^{\mathbb{Q}}\left[U_T \mathbb{1}_{V_S > \gamma} \right]\right]$$

$$= \mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{1}_{V_S > \gamma}\, \mathbb{E}_S^{\mathbb{Q}}\left[U_T \right]\right]$$

$$= \mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{1}_{V_S > \gamma}\, U_S \right]$$

$z, z^{\perp} \overset{\mathbb{Q}}{\sim} N(0,1)$ independent

now, $U_S \overset{d}{=} U_0\, e^{-\frac{1}{2}\sigma^2 S + \sigma\sqrt{S}\,(\rho z + \sqrt{1-\rho^2}\, z^{\perp})}$

$V_S \overset{d}{=} V_0\, e^{-\frac{1}{2}\eta^2 S + \eta\sqrt{S}\, z}$

$$\therefore\ c_t = U_0\, e^{-\frac{1}{2}\sigma^2 S}\, \mathbb{E}^{\mathbb{Q}}\Big[\underbrace{e^{\sigma\sqrt{S(1-\rho^2)}\, z^{\perp}}\quad e^{\sigma\sqrt{S}\,\rho z}\, \mathbb{1}_{z > -z^*}}_{\text{independent}} \Big]$$

here $z^* = \dfrac{\ln(V_0/\gamma) - \frac{1}{2}\eta^2 S}{\eta\sqrt{S}}$

$$= U_0\, e^{-\frac{1}{2}\sigma^2 S}\, \mathbb{E}^{\mathbb{Q}}\left[e^{\sigma\sqrt{S(1-\rho^2)}\, z^{\perp}} \right] \mathbb{E}^{\mathbb{Q}}\left[e^{\sigma\sqrt{S}\,\rho z}\, \mathbb{1}_{z > -z^*} \right]$$

$$= u_0 \, e^{-\frac{1}{2}\sigma^2 s} \cdot e^{\frac{1}{2}\sigma^2(1-\rho^2)s} \cdot \int_{-z^*}^{\infty} e^{\sigma\rho\sqrt{s}\,z} \, e^{-\frac{1}{2}z^2} \, \frac{dz}{\sqrt{2\pi}}$$

$$= u_0 \, e^{-\frac{1}{2}\rho^2\sigma^2 s} \int_{-z^*}^{\infty} e^{-\frac{1}{2}(z-\sigma\rho\sqrt{s})^2 + \frac{1}{2}\sigma^2\rho^2 s} \, \frac{dz}{\sqrt{2\pi}}$$

$$= u_0 \, \Phi(z^* + \sigma\rho\sqrt{s})$$

Q8

Thursday, November 19, 2009
2:05 PM

8. *[10]* Prove that

$$\int_0^t W_s dZ_s + \int_0^t Z_s dW_s = W_t Z_t - \rho t \quad a.s.$$

Do not use Ito's lemma, but rather use the fundamental definition of the stochastic integrals.

consider $A \triangleq \int_0^t W_s dZ_s + \int_0^t Z_s dW_s - W_t Z_t + \rho t$

$$A = \lim_{\|\pi\| \downarrow 0} \sum_k A_k$$

$$A_k \triangleq W_{t_{k-1}} \Delta Z_k + Z_{t_{k-1}} \Delta W_k - (W_{t_k} Z_{t_k} - W_{t_{k-1}} Z_{t_{k-1}}) + \rho \Delta t_k$$

$$= -\Delta W_k \Delta Z_k + \rho \Delta t_k$$

Note: $\mathbb{E}\left[\sum_k A_k\right] = \sum_k \mathbb{E}[A_k] = 0$

$$\mathbb{V}\left[\sum_k A_k\right] = \sum_k \mathbb{V}[A_k] = \sum_k \mathbb{V}[\Delta W_k \Delta Z_k]$$

$$= \sum_k \mathbb{E}\left[(\Delta W_k)^2 (\Delta Z_k)^2\right]$$

$$= \sum_k \mathbb{E}\left[(N_1 \sqrt{\Delta t_k})^2 (\rho N_1 \sqrt{\Delta t_k} + \sqrt{1-\rho^2} N_2 \sqrt{\Delta t_k})^2\right]$$

$$= \sum_k (\Delta t_k)^2 \rho$$

$$\leq \rho \|\pi\| \sum_k \Delta t_k = \rho \|\pi\| t$$

$$\xrightarrow[\|\pi\| \downarrow 0]{} 0$$

$\therefore \sum_k A_k \to 0$ a.s. $\quad \therefore A = 0$ a.s.

$$\therefore \quad \int_0^t W_s \, dZ_s + \int_0^t Z_s \, dW_s = W_t Z_t - \rho t \quad a.s.$$

Q9

Thursday, November 19, 2009

2:06 PM

9. Consider the Vasicek model for the short rate of interest:

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dW_t$$

where W_t is a $\mathbb{Q}$-Wiener process. The solution to this SDE is

$$r_s = \theta + (r_t - \theta)\,e^{-\kappa(s-t)} + \sigma \int_t^s e^{-\kappa(s-u)} dW_u \qquad \text{for} \quad t \le s.$$

(a) *[5]* Show that the distribution of $I_t^T = \int_t^T r_s\,ds$ is normal with mean m and variance v with

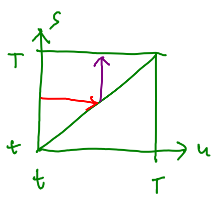
$$m = \theta((T-t) - B(T-t;\kappa)) + B(T-t;\kappa)\,r_t\,,$$
$$v = \frac{\sigma^2}{\kappa^2}\left((T-t) + B(T-t;2\kappa) - 2B(T-t;\kappa)\right)$$

where, $B(\tau;\kappa) = \frac{1}{\kappa}(1 - e^{-\kappa\tau})$.

$$I = \int_t^T r_s\,ds$$
$$= \theta(T-t) + (r_t - \theta)\overbrace{\int_t^T e^{-\kappa(s-t)}}^{B} ds + \sigma \underbrace{\int_t^T \left(\int_t^s e^{-\kappa(s-u)} dW_u\right) ds}_{A}$$

$$B = \int_t^T e^{-\kappa(s-t)} ds = \frac{1 - e^{-\kappa(T-t)}}{\kappa} = B(T-t;\kappa)$$

$$A = \int_t^T \int_t^s e^{-\kappa(s-u)} dW_u\,ds = \int_t^T \left(\int_u^T e^{-\kappa(s-u)} ds\right) dW_u$$



Change order of integration from → to ↑

$$\Rightarrow A = \int_t^T \left(\frac{1 - e^{-\kappa(T-u)}}{\kappa}\right) dW_u \sim \mathcal{N}(0\,;\,v^2)$$

$$v^2 = \mathbb{E}\left[\left(\int_t^T \frac{1 - e^{-\kappa(T-u)}}{\kappa} dW_u\right)^2\right]$$
$$= \mathbb{E}\left[\int_t^T \left(\frac{1 - e^{-\kappa(T-u)}}{\kappa}\right)^2 du\right]$$
$$= \frac{1}{\kappa^2}\int_t^T \left(1 + e^{-2\kappa(T-u)} - 2e^{-\kappa(T-u)}\right) du$$

$$= \frac{1}{\kappa^2}\left((T-t) - 2\,\frac{1-e^{-\kappa(T-t)}}{\kappa} + \frac{1-e^{-2\kappa(T-t)}}{2\kappa} \right)$$

$$= \frac{1}{\kappa^2}\left[(T-t) - 2B(T-t;\kappa) + B(T-t;2\kappa) \right]$$

and so $I \sim N(m; v^2)$ as required.

(b) *[5]* Show that price of a T-maturity bond satisfies the SDE

$$\frac{dP_t(T)}{P_t(T)} = r_t\,dt - \sigma\,B(T-t;\kappa)\,dW_t\,.$$

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds} \,\middle|\, r_t \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{-I_t^T} \,\middle|\, r_t \right]$$

$$= \exp\left\{ -m + \tfrac{1}{2}v^2 \right\}$$

$$= \exp\left\{ h(t) - B(T-t;\kappa)\,r_t \right\}$$

$$h(t) = \tfrac{1}{2}v^2 + \theta\left((T-t) - B(T-t;\kappa) \right)$$

and so,

$$dP_t(T) = \underbrace{\left(\partial_t P + \kappa(\theta - r)\,\partial_r P + \tfrac{1}{2}\sigma^2\,\partial_{rr} P \right)}_{\mu^P} dt + \partial_r P\;\sigma\,dW_t$$

$\hookrightarrow -B(T-t;\kappa)\;P_t(T)$

but since P is traded $\Rightarrow \mu^P = r_t\,P_t(T)$
(i.e. traded assets grow at the risk-free rate)

$$\Rightarrow \frac{dP_t(T)}{P_t(T)} = r_t P_t(T)\,dt - B(T-t;\kappa)\,\sigma\,dW_t$$