Thursday, October 02, 2008
1:27 PM

1. *[10]* Please indicate true or false. **no explanations required**

   -1 for incorrect answer, $+2$ for correct answer, 0 for blank answer .

   (a) [T] [F]

   If an economy admits a strategy which costs nothing and at a future time has strictly positive values with probability greater than zero, then this economy admits an arbitrage.

   must also true $\mathbb{P}(V_t \geq 0) = 1$

   (b) [T] [F]

   The price of a put option always increases with volatility.

   more uncertainty increases value of the option

   (c) [T] [F]

   In a one-period economy, the risk-neutral branching probabilities are always uniquely determined.

   only if number of traded assets equal to number of states, and the assets are not redundant.

   (d) [T] [F]

   If S_t is the price of a traded stock, then in the Black-Scholes economy, the risk-neutral expected rate of return of S_t^2 is equal to $2r + \sigma^2$.

$$\mathbb{E}^{\mathbb{Q}}[S_t^2] = S_0 e^{2(r - \frac{1}{2}\sigma^2)t} \, \mathbb{E}^{\mathbb{Q}}[e^{2\sigma\sqrt{t} Z}]$$

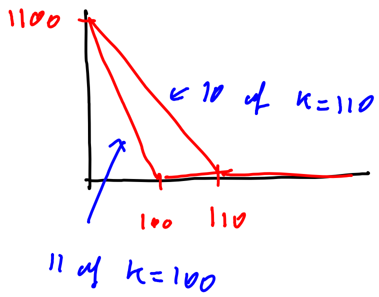
$$= S_0 e^{2(r - \frac{1}{2}\sigma^2)t} \, e^{\frac{1}{2} 4\sigma^2 t}$$

$$= S_0 e^{(2r + \sigma^2)t}$$

(e) [T] [F]

A put option struck at \$100 trades at \$10, while a put option struck at \$110 trades at \$11. Both puts have the same time to maturity. This economy admits an arbitrage.

[Hint: Consider 11 units of the first put and 10 units of the second put]



$\therefore\ 10\,P_{110} > 11\,P_{100}$ to avoid arbitrage

but $10 \times 11 = 110$

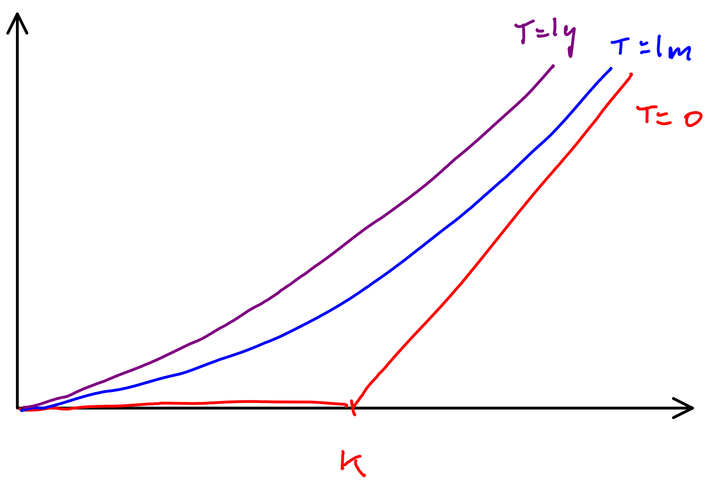
and $11 \times 10 = 110$

$\therefore\ \exists$ an arbitrage.

2. Sketch the option price as a function of the current spot-level for maturities of $T = 0$, $T = 1$ month and $T = 1$ year for

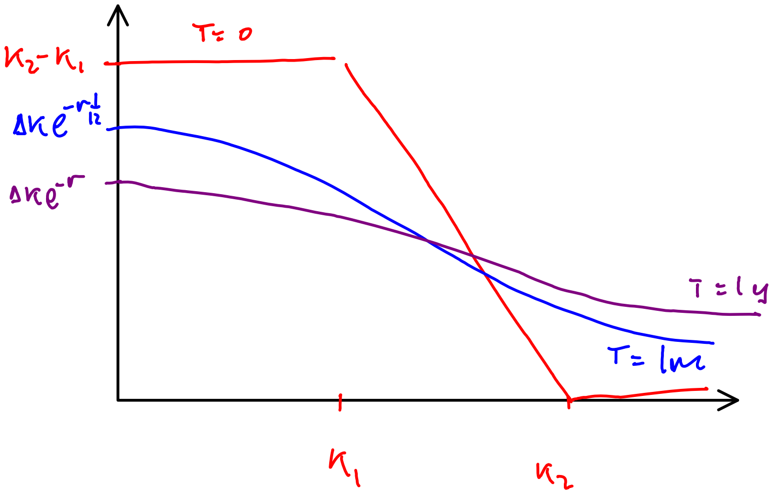
(a) *[5]* call option

[draw the three curves on the same graph, clearly label them and any interesting points.]

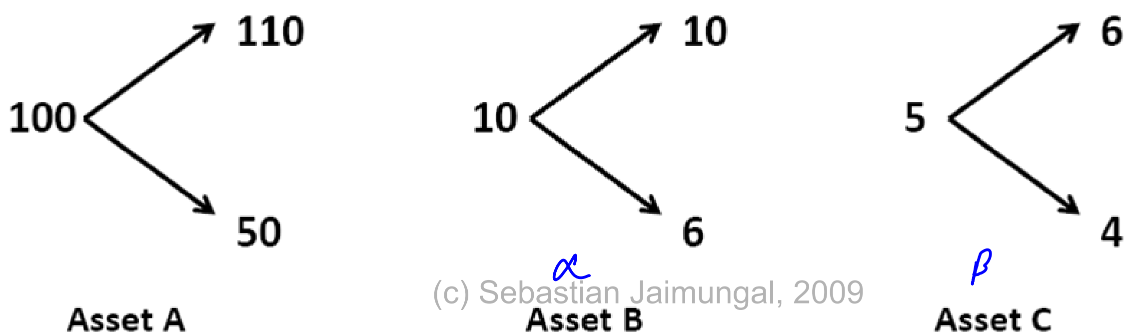


(b) *[5]* bear spread option. This option can be viewed as a long put struck at K_2 and a short put struck at K_1 $(0 < K_1 < K_2 < \infty)$

[draw the three curves on the same graph, clearly label them and any interesting points..]



3. *[10]* Consider an economy with the three traded assets below. Construct an arbitrage strategy.



$10\alpha + 6\beta = 110$

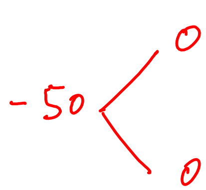
$6\alpha + 4\beta = 50 \Rightarrow 9\alpha + 6\beta = 75$

$\therefore \alpha = 35$ & $\beta = -40$

and long 35 B + short 40 C

has value 150 ∴ asset A is too cheap

— long A, short 35, long 40 C

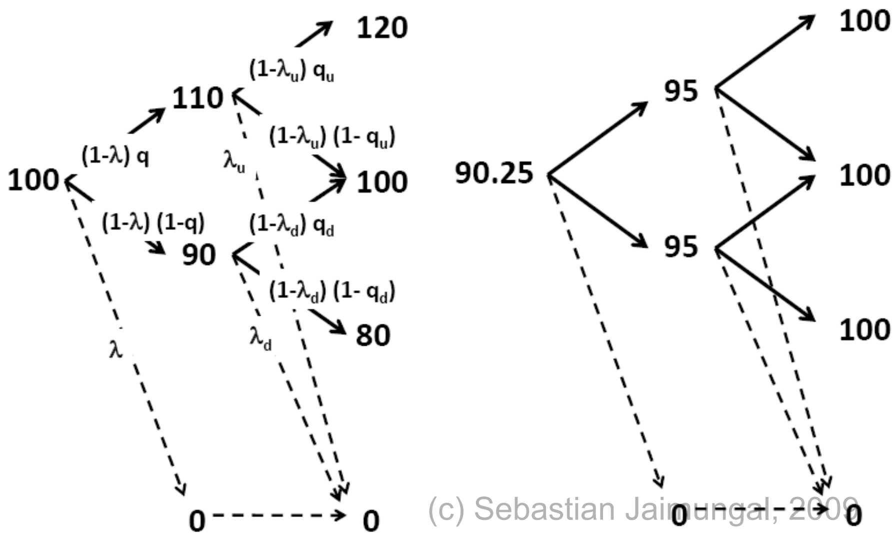


long 5 more B

$$0 \begin{cases} 50 \\ 30 \end{cases}$$

an arbitrage portfolio!

4. Consider an economy with a defaultable stock and bond (interest rates are 0 in all states of the world):



Defaultable Stock price tree

Defaultable Bond price tree

(a) Determine the risk-neutral default probabilities shown in the diagram $a_u, a_d, b_u, b_d, c_u, c_d$. *[Hint: It is easiest if you find b_d and c_d by using the default bond tree, then find b_u and c_u from the default stock tree, and finally find a_d and a_u from both trees.]*

from default bond:

$$95 = (1-\lambda_u)100 \Rightarrow \lambda_u = 1/20$$
$$95 = (1-\lambda_d)100 \Rightarrow \lambda_d = 1/20$$
$$90.25 = (1-\lambda)\,100 \Rightarrow \lambda = 1/20$$

from default stock:

$$(1-\lambda_u)\,(q_u 120 + (1-q_u)100) = 110$$

$$\Rightarrow \quad 100 + 20 q_u = 110 \cdot \frac{20}{19}$$

$$\Rightarrow q_u = \frac{110}{19} - 5 = \frac{15}{19}$$

$$(1-\lambda_d)(q_d\, 100 + (1-q_d)\, 80) = 90$$

$$\Rightarrow \quad 80 + 20 q_d = 90 \cdot \frac{20}{19}$$

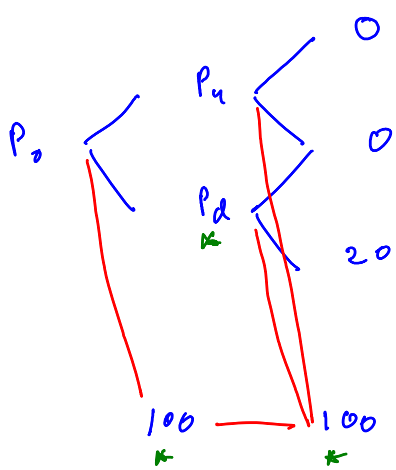
$$\Rightarrow \quad q_d = \frac{90}{19} - 4 = \frac{14}{19}$$

$$(1-\lambda)(q\, 110 + (1-q)\, 90) = 100$$

$$\Rightarrow \quad 90 + 20q = 100 \cdot \frac{20}{19}$$

$$\Rightarrow \quad q = \frac{100}{19} - \frac{9}{2} = \frac{29}{38}$$

(b) Value an American put option with the stock as underlier, strike of 100 and maturity of two periods.



* = exercise nodes

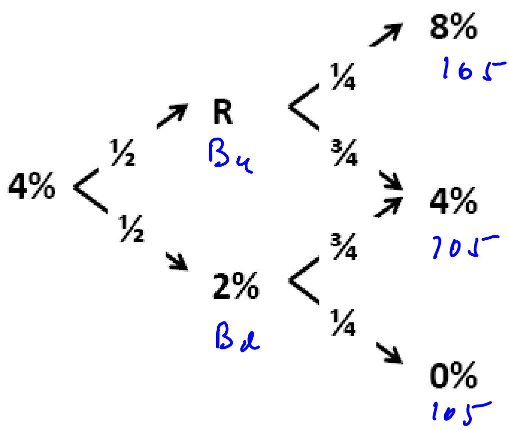
$$P_u^* = \frac{1}{20} \times 100 = 5 > I_u = 0 \quad \therefore \quad P_u = 5$$

$$P_d^* = \left(\frac{1}{20} \times 100 + \frac{19}{20} \cdot \frac{4}{19} \cdot 20\right) \frac{95}{100} = 8.55 < I_u = 10 \quad \therefore \quad P_d = 10$$

$$P_0 = \left(\frac{1}{20}\times 100 + \frac{19}{20}\left(\frac{29}{38}\cdot 5 + \frac{9}{38}\cdot 10\right)\right)\frac{90.25}{95}$$

$$= 10.33$$

5. Consider the interest rate tree shown in the diagram below. The rates correspond to effective discounting – e.g. discounting over the first period is $1/(1+0.04)$. The probabilities shown are risk-neutral probabilities.



(a) *[5]* Consider a 2-year coupon bearing bond with coupons of \$5 paid every year and notional of \$100. Determine the rate R such that the bond is valued at par (i.e. has current value of 100).

$$B_u = \frac{105}{1+R} + 5, \qquad B_d = \frac{105}{1.02} + 5$$

$$B_0 = \frac{1}{1.04}\left[\frac{1}{2}\left(\frac{105}{1+R} + 5\right) + \frac{1}{2}\left(\frac{105}{1.02} + 5\right)\right]$$

$$= 54.299 + \frac{50.48}{1+R} = 100$$

$$\Rightarrow \quad R = 10.46\,\%$$

(b) *[5]* Determine the price and replication strategy of a call option maturing at $t = 1$ written on the coupon bearing bond with strike equal to today's price of the bond. Note: the option holder will not receive the coupon due at $t = 1$.

$$C_0 \begin{cases} (B_u - 5 - 100)_+ = 0 \\ (B_d - 5 - 100)_+ = 2.94 \end{cases}$$

$$C_0 = \frac{2.94 \times \frac{1}{2}}{1.04} = 1.41$$

6. (a) *[5]* Assuming the Black-Scholes model, derive an expression for a contingent claim which pays the geometric average of the asset price at two points in time. That is, the claim pays $(S_{T_1} S_{T_2})^{\frac{1}{2}}$ at maturity T_2 where $0 < T_1 < T_2$.

$$V_0 = e^{-rT_2} \mathbb{E}^{Q}[(S_{T_1} S_{T_2})^{1/2}]$$

$$S_{T_1} = S_0 \, e^{(r - \frac{1}{2}\sigma^2)T_1 + \sigma\sqrt{T_1} Z_1}$$

$$S_{T_2} = S_{T_1} \, e^{(r - \frac{1}{2}\sigma^2)(T_2 - T_1) + \sigma\sqrt{T_2 - T_1} Z_2}$$

$$= S_0 \, e^{(r - \frac{1}{2}\sigma^2)T_2 + \sigma\sqrt{T_1} Z_1 + \sigma\sqrt{T_2 - T_1} Z_2}$$

$$Z_1, Z_2 \text{ iid} \underset{Q}{\sim} N(0,1)$$

$$\Rightarrow V_0 = S_0 \, e^{-rT_2 + \frac{1}{2}(r - \frac{1}{2}\sigma^2)(T_1 + T_2)}$$

$$\times \mathbb{E}^{Q}[e^{\sigma\sqrt{T_1} Z_1} \; e^{\frac{1}{2}\sigma\sqrt{T_2 - T_1} Z_2}]$$

$$= S_0 \, e^{-rT_2 + \frac{1}{2}(r - \frac{1}{2}\sigma^2)(T_1 + T_2)}$$

$$\times e^{\frac{1}{2}\sigma^2 T_1} \times e^{\frac{1}{2}\cdot\frac{1}{4}\sigma^2 (T_2 - T_1)}$$

$$= S_0 \, e^{\frac{1}{2} r (T_1 - T_2) + \frac{1}{8}(T_1 + T_2)\sigma^2}$$

(b) *[5]* Assuming the Black-Scholes model, derive an expression for a "forward start digital call option". A forward start digital call is an option which pays 1 at the maturity date T if the stock price at time T is larger than the stock price at time U. ($U < T$) Write your answer in terms of $\Phi(x) := \mathbb{Q}(Z < x)$ where Z is a standard normal random variable under the measure $\mathbb{Q}$.

$$V_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{S_T > S_u}\right]$$

now $\quad S_T = S_u\, e^{(r-\frac{1}{2}\sigma^2)(T-u) + \sigma\sqrt{T-u}\,Z}$

$$\Rightarrow V_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{e^{(r-\frac{1}{2}\sigma^2)(T-u)+\sigma\sqrt{T-u}\,Z} > 1}\right]$$

$$= e^{-rT}\,\mathbb{Q}\left(Z > -\frac{(r-\frac{1}{2}\sigma^2)}{\sigma}\sqrt{T-u}\right)$$

$$= e^{-rT}\,\Phi\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma}\sqrt{T-u}\right)$$

7. Suppose you model stock prices in a CRR like fashion. However, you assume that

$$S_n = S_{n-1}\exp\{(r - \tfrac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\; x_n\}$$

where $x_1, x_2, \ldots$ are iid r.v. with $\mathbb{P}(x_1 = +1) = p$ and $\mathbb{P}(x_1 = -1) = 1 - p$.

(a) *[5]* Prove that if we force

$$\mathbb{E}^{\mathbb{P}}[S_T] = S_0\, e^{\mu T}\ ,$$
$$\mathbb{V}^{\mathbb{P}}[\ln(S_T/S_0)] = \sigma^2\, T$$

in the limit as $\Delta t \downarrow 0$ while $T = n\Delta t$ is held fixed. Then,

$$p = \frac{1}{2}\left(1 + \frac{\mu - r}{\sigma}\sqrt{\Delta t}\right) + O(\Delta t)\ .$$

$$S_T = S_0\, e^{(r-\frac{1}{2}\sigma^2)\Delta t\, n + \sigma\sqrt{\Delta t}\,(x_1 + \cdots + x_n)}$$

$$= S_0\, e^{(r-\frac{1}{2}\sigma^2)T + X}$$

where $X = \sigma\sqrt{\Delta t}\,(x_1 + \cdots + x_n) \xrightarrow[n\to\infty]{} \mathcal{N}(m, v^2)$

$$m = \sigma\, n \sqrt{\Delta t}\,(2p-1)$$

$$v^2 = \sigma^2 \Delta t\, n\,(1 - (2p-1)^2)$$

$$= \sigma^2 T\,(1 - (2p-1)^2)$$

$$\therefore \mathbb{E}^{\mathbb{P}}[S_T] = S_0\, e^{(r - \frac{1}{2}\sigma^2)T + \frac{1}{2}v^2 + m}$$

$$\therefore (r - \tfrac{1}{2}\sigma^2)T + \tfrac{1}{2}v^2 + m \overset{\text{require}}{=} \mu T$$

$$\Rightarrow \quad m + \tfrac{1}{2}v^2 = ((\mu - r) + \tfrac{1}{2}\sigma^2)\,T$$

$$\mathbb{V}^{\mathbb{P}}[\ln(S_T/S_0)] = v^2 \overset{\text{require}}{=} \sigma^2 T$$

$$\therefore \quad m = ((\mu - r) + \tfrac{1}{2}\sigma^2)\,T - \tfrac{1}{2}\sigma^2 T$$

$$= (\mu - r)\,T$$

$$\therefore \quad \sigma\, n\sqrt{\Delta t}\,(2p-1) = (\mu - r)\,T$$

$$\Rightarrow \quad p = \frac{1}{2}\left(1 + \frac{\mu - r}{\sigma}\sqrt{\Delta T}\right)$$

(b) *[5]* Prove that, in the limit as $\Delta t \downarrow 0$ while $T = n\Delta t$ is held fixed, the risk neutral probability in this model is (with constant rate of interest r)

$$q = \frac{1}{2} + O(\Delta t)\ .$$

and that

$$\mathbb{E}^{\mathbb{Q}}[S_T] = S_0\, e^{rT}\ ,$$
$$\mathbb{V}^{\mathbb{Q}}[\ln(S_T/S_0)] = \sigma^2\, T\ .$$

$$S \begin{cases} S\, e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} \\ S\, e^{(r-\frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}} \end{cases}$$

$$\Rightarrow \quad S = e^{-r\Delta t}\Big[\, q\, S e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} + (1-q)\, S e^{(r-\frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}} \,\Big]$$

$$\Rightarrow \quad q = \frac{e^{r\Delta t} - e^{(r-\frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}}{e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} - e^{(r-\frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}}$$

$$\simeq \frac{(\cancel{1} + \cancel{r\Delta t} + \cdots) - (\cancel{1} + (\cancel{r} - \cancel{\tfrac{1}{2}\sigma^2})\Delta t - \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2}\Delta t + \cdots)}{\left\{(\cancel{1} + \cancel{(r-\tfrac{1}{2}\sigma^2)}\Delta t + \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2\Delta t} + \cdots) - (\cancel{1} + (r - \cancel{\tfrac{1}{2}\sigma^2})\Delta t - \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2\Delta t} + \cdots)\right\}}$$

$$= \frac{\sigma\sqrt{\Delta t} + \cdots}{2\sigma\sqrt{\Delta t} + \cdots} = \frac{1}{2} + \cdots$$