Q1

Tuesday, October 20, 2009
10:53 AM

1. *[10]* Please indicate true or false. **no explanations required**

   −1 for incorrect answer, +2 for correct answer, 0 for blank answer .

   (a) [T] [F]

   All two period, two state (binomial) economies are arbitrage free.

   False . e.g. $1 \begin{cases} 1 \\ 1 \end{cases}$ $\quad 1 \begin{cases} 2 \\ 2 \end{cases}$

   (b) [T] [F]

   The price of a call option always decreases with increasing volatility.

   False. it increases.

   (c) [T] [F]

   If the branching probabilities are unique, then all contingent claims can be replicated.

   true. unique q's $\Rightarrow$ unique prices $\Rightarrow$ replication

   (d) [T] [F]

   The risk-neutral return of the short rate of interest in a stochastic interest rate model is equal to r.

   short rate of interest is not traded so $\mathbb{E}^{Q}[r_T] \neq r_T e^{r}$

   (e) [T] [F]

   Suppose interest rates are zero. An at-the-money put option is worth 0.80 and a call option with the same strike and maturity is worth 0.75. This economy admits an arbitrage.

   [At-the-money means the strike equals the spot.]

   Call − Put = Spot − K = 0 since at the

money

$\therefore$ arbitrage.

Q2

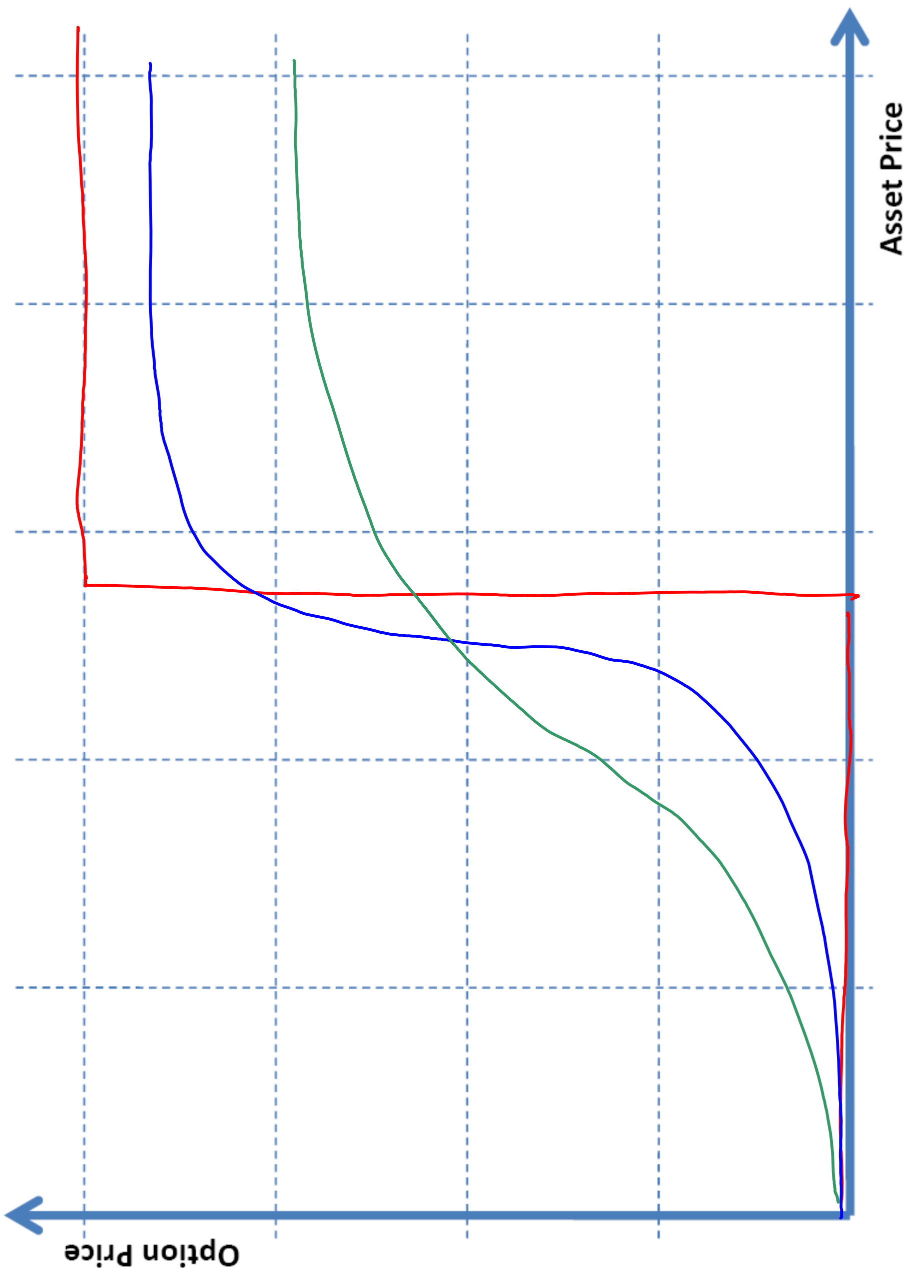
Tuesday, October 20, 2009

2:14 PM

2. Sketch the option price as a function of the current spot-level for maturities of $T = 0$, $T = 1$ month and $T = 1$ year for

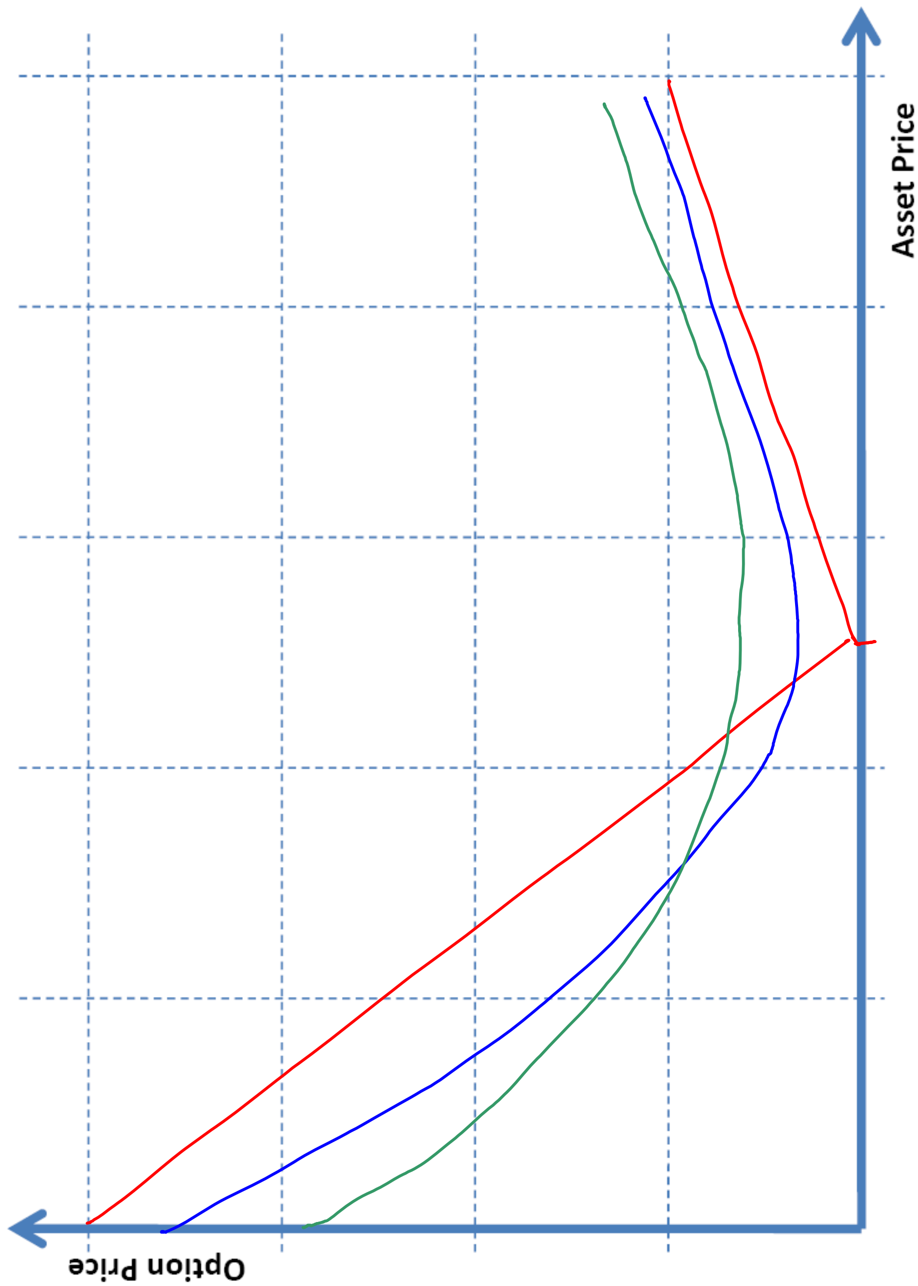
   (a) *[5]* digital call option (which pays 1 if $S > K$ and 0 otherwise).

   [draw the three curves on the same graph, clearly label them and any interesting points.]



(b) *[5]* A portfolio of 4 long puts and 1 long call, both struck at $1.

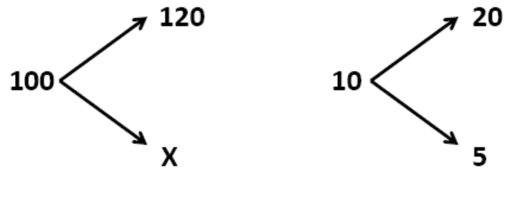
[draw the three curves on the same graph, clearly label them and any interesting points..]



Q3

Tuesday, October 20, 2009
2:18 PM

3. *[10]* Consider an economy with the two traded assets below. Find the values of X such that the economy is free of arbitrage.



Asset A   Asset B

Method I:

$$100(1+r) = 120q + X(1-q)$$

$$10(1+r) = 20q + 5(1-q)$$

$$\Rightarrow \quad 10 = \frac{120q + X(1-q)}{5+15q}$$

$$\Rightarrow \quad 50 + 150q = (120 - X)q + X$$

$$\Rightarrow \quad (30+X)q = X - 50$$

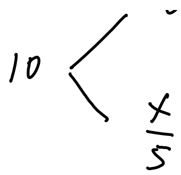
$$\Rightarrow \quad q = \frac{X-50}{30+X}$$

$$0 < q < 1 \iff \boxed{X > 50}$$

Method 2:

Choose B as numeraire, so relative price tree is

b



$$10 = 6\,q^B + \frac{x}{5}\,(1-q^B)$$

$$50 = 30\,q^B + x\,(1-q^B)$$

$$\Rightarrow \quad q^B = \frac{50-x}{30-x}$$

to avoid arbitrage need $\quad 0 < q^B < 1$

$$\Rightarrow \quad 0 < \frac{50-x}{30-x} < 1$$

① $x < 30 \Rightarrow \quad 0 < 50 - x < 30 - x$

$\Rightarrow \quad x < 50$ and $50 < 30$ contradiction

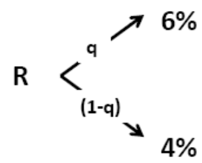
② $x > 30 \Rightarrow \quad 0 > 50 - x > 30 - x$

$$\Rightarrow \quad \boxed{x > 50}$$

Q4

Tuesday, October 20, 2009
2:30 PM

4. Consider the interest rate tree shown in the diagram below – each time step is 1-year. The rates correspond to effective discounting – e.g. discounting over the first period is $1/(1+R)$. The probabilities shown are risk-neutral probabilities.



(a) *[6]* The price of a one-year bond on a notional of \$100 is \$95.2381. As well, a 2-year coupon bearing bond with coupons of \$5 paid every year and notional of \$100 is valued at par. Calibrate this model to the market prices, i.e. determine R and q such that the market prices are equal to the model prices.

$$P(1) = \frac{100}{1+R} = 95.2381 \Rightarrow R = 5\%$$

$$P(2) \quad \left(\frac{105}{1.06} + 5\right) = 104.06$$

$$\quad\quad\;\; \left(\frac{105}{1.04} + 5\right) = 105.96$$

$$100 = P(2) = \frac{1}{1.05}\left[104.06\, q + 105.96\,(1-q)\right]$$

$$\Rightarrow \quad 105 = 104.06\, q + 105.96\,(1-q)$$

$$\Rightarrow \quad q = \frac{0.96}{105.96 - 104.06} = 0.5053$$

(b) *[4]* Now assume that $q = \frac{1}{2}$ and $R = 5\%$. As well, you can only trade using the 1-year and 2-year **zero coupon** bonds with notionals of \$100 (i.e. 1-year zero coupon bond pays \$100 at year 1, and the 2 year zero coupon bond pays \$100 at year 2).

What is the replication strategy of an option which pays \$100 if the interest rate drops to 4%?

$$95.2381 \left\langle \begin{matrix} 100 \\ 100 \end{matrix} \right.$$

α - units

$$P(2) \left\langle \begin{matrix} 94.34 = \frac{100}{1.06} \left\langle \begin{matrix} 100 \\ 100 \end{matrix} \right. \\ 96.15 = \frac{100}{1.04} \left\langle \begin{matrix} 100 \\ 100 \end{matrix} \right. \end{matrix} \right.$$

β - units

$$P(2) = \frac{1}{1.05}\left[\frac{1}{2}94.34 + \frac{1}{2}96.15\right] = 90.71$$

$$C \left\langle \begin{matrix} 0 = \alpha 100 + \beta 94.34 \\ 100 = \alpha 100 + \beta 96.15 \end{matrix} \right.$$

$$\Rightarrow \quad \beta = \frac{100}{96.15 - 94.34} = +55.25$$

$$\alpha = -\beta \frac{96.15}{100} = -53.12$$

Q5

Tuesday, October 20, 2009
2:49 PM

5. Assume an equity price S_t is modeled as in the Black-Scholes model (i.e. the limiting case of the CRR model as $\Delta t \downarrow 0$ and interest rates are constant at r). For each of the following, write your answers terms of $\Phi(x) \triangleq \mathbb{Q}(Z < x)$ where Z is a standard normal random variable under the risk-neutral measure $\mathbb{Q}$.

   (a) *[5]* Derive an expression for the $(t = 0)$ price of an option with T-maturity payoff

$$\varphi = \min(S_T\ ;\ K)\ .$$

   Here K is a constant.

$$S_T \stackrel{d}{=} S \exp\left\{ (r - \tfrac{1}{2}\sigma^2) T + \sigma\sqrt{T}\, Z \right\}, \quad Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$V = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[\min(S_T, K) \right]$$

then, $\min(S_T, K) = K\, \mathbb{1}_{S_T > K} + S_T\, \mathbb{1}_{S_T \leq K}$

$$\therefore\ V = e^{-rT}\left(K\, \mathbb{Q}(S_T > K) + \mathbb{E}^{\mathbb{Q}}\left[S_T\, \mathbb{1}_{S_T \leq K} \right] \right)$$

now $\mathbb{E}^{\mathbb{Q}}\left[S_T\, \mathbb{1}_{S_T \leq K} \right]$

$$= \int_{-\infty}^{\infty} S\, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} z}\, \mathbb{1}_{z \leq z^*}\, e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}$$

where $$z^* = -\frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$= S\, e^{(r-\frac{1}{2}\sigma^2)T} \int_{-\infty}^{z^*} e^{\sigma\sqrt{T}z - \frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}$$

$$= S\, e^{(r-\frac{1}{2}\sigma^2)T} \int_{-\infty}^{z^*} e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2 + \frac{1}{2}\sigma^2 T} \frac{dz}{\sqrt{2\pi}}$$

$$= S\, e^{rT} \int_{-\infty}^{z^*-\sigma\sqrt{T}} e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}$$

$$= S\, e^{rT}\, \Phi\Big(\underbrace{- \frac{\ln(S/K) + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{d_+}\Big)$$

as well,

$$\mathbb{Q}(S_T > K) = \mathbb{Q}(Z > z^*) = \Phi(-z^*)$$

$$= \Phi\Big(\underbrace{\frac{\ln(S/K) + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{d_-}\Big)$$

So Final answer is:

$$V = K e^{-rT}\, \Phi(d_-) + S\, \Phi(-d_+)$$

(b) *[5]* Derive an expression for the $(t = 0)$ price of a forward start option with T-maturity payoff

$$\varphi = \min(S_T\ ;\ k\ S_U)\ .$$

Here, $0 < U < T$ and k is a proportionality constant.

$$V = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[\min(S_T \,;\, k\, S_u) \right]$$

$$= e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\min(S_T , k\, S_u) \mid S_u \right] \right]$$

$$= e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\Big[\Big(k\, S_u\, e^{-r(T-u)}\, \Phi(\ell_-) + S_u\, \Phi(-\ell_+) \Big)\, e^{r(T-u)} \Big]$$

here
$$\ell_\pm = \frac{\ln(\cancel{S_u}/k\,\cancel{S_u}) + (r \pm \tfrac{1}{2}\sigma^2)(T-u)}{\sigma\sqrt{T-u}}$$

are constants!

$$\Rightarrow V = \Big(k\, e^{-r(T-u)}\, \Phi(\ell_-) + \Phi(-\ell_+) \Big)\, e^{-ru}\, \mathbb{E}^{\mathbb{Q}}[S_u]$$

$$= \Big(k\, e^{-r(T-u)}\, \Phi(\ell_-) + \Phi(-\ell_+) \Big)\, S$$

Q6

Tuesday, October 20, 2009
3:28 PM

6. Consider the CRR model of stock prices

$$S_{n\Delta t} = S_{(n-1)\Delta t} \exp\{\sigma \sqrt{\Delta t}\, x_n\}$$

where $x_1, x_2, \ldots$ are iid r.v. with $\mathbb{P}(x_1 = +1) = p$ and $\mathbb{P}(x_1 = -1) = 1 - p$. Interest rates are constant so that the money-market account M_t evolves as

$$M_{n\Delta t} = M_{(n-1)\Delta t} \exp\{r\Delta t\}$$

(a) *[6]* Prove that under the measure induced by using S as a numeraire asset (call this measure $\mathbb{Q}_S$), as $\Delta t \downarrow 0$ one has

$$S_T \stackrel{d}{=} S \exp\{(r + \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z\}$$

where $Z \stackrel{\mathbb{Q}_S}{\sim} \mathcal{N}(0,1)$.

[Note that the drift is $r + \frac{1}{2}\sigma^2$ and NOT $r - \frac{1}{2}\sigma^2$ as it is under the risk-neutral measure $\mathbb{Q}$.]

use S as numeraire, so relative price tree is

$$1/S \;\begin{array}{l} \nearrow^{q_s}\; e^{r\Delta t} / S e^{\sigma\sqrt{\Delta t}} \\[1em] \searrow\; e^{r\Delta t} / S e^{-\sigma\sqrt{\Delta t}} \end{array}$$

$$1 = q_s\, e^{r\Delta t - \sigma\sqrt{\Delta t}} + (1 - q_s)\, e^{r\Delta t + \sigma\sqrt{\Delta t}}$$

$$\Rightarrow \quad q_s = \frac{e^{-r\Delta t} - e^{\sigma\sqrt{\Delta t}}}{e^{-\sigma\sqrt{\Delta t}} - e^{\sigma\sqrt{\Delta t}}}$$

want q_s as $\Delta t \downarrow 0$ ...

$$q_s = \frac{(1 - r\Delta t) - (1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + \cdots}{(1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) - (1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + \cdots}$$

$$= \frac{-\sigma\sqrt{\Delta t} - (r + \frac{1}{2}\sigma^2)\Delta t + \cdots}{}$$

$$- 2\sigma\sqrt{\Delta t} + \cdots$$

$$= \frac{1}{2}\left[1 + \frac{r + \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t} + \cdots \right]$$

so then since

$$\ln(S_T/S) = \sigma\sqrt{\Delta t}\sum_{m=1}^{n} x_m$$

$$\xrightarrow[\Delta t \downarrow 0]{} \mathcal{N}(\text{mean}, \text{var}) \quad \text{by CLT}$$

$$\mathbb{E}^{\mathbb{Q}^S}[\ln(S_T/S)] = \sigma\sqrt{\Delta t}\, n\, \mathbb{E}^{\mathbb{Q}^S}[x_1]$$

$$= \sigma\sqrt{\Delta t}\, n\,(2q_S - 1)$$

$$= \sigma\sqrt{\Delta t}\, n \cdot \left(\frac{r + \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t} + \cdots \right)$$

$$\longrightarrow (r + \tfrac{1}{2}\sigma^2)\underbrace{n\Delta t}_{\to T}$$

and,

$$\mathbb{V}^{\mathbb{Q}^S}[\ln(S_T/S)] = \sigma^2\Delta t\, n\, \mathbb{V}^{\mathbb{Q}^S}[x_1]$$

$$= \sigma^2\Delta t\, n\left(1 - (2q_S - 1)^2 \right)$$

$$= \sigma^2\Delta t\, n\left(1 - \left(\frac{r + \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t} + \cdots \right)^2 \right)$$

$$\longrightarrow \sigma^2 T$$

$$\therefore \quad S_T \stackrel{d}{=} S\exp\{(r + \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z\}$$

$$Z \underset{\mathbb{Q}^S}{\sim} \mathcal{N}(0,1)$$

(b) *[4]* Using the measures $\mathbb{Q}_S$ and $\mathbb{Q}$, show that the $(t=0)$ price of a T-maturity put option is

$$K\ e^{-rT}\ \Phi(-d_-) - S\ \Phi(-d_+), \qquad d_\pm = \frac{\ln(S/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\ .$$

YOU ARE NOT ALLOWED TO COMPUTE INTEGRALS IN THIS QUESTION.

[Hint: Write the put payoff in terms of a digital option $K\mathbb{I}_{S_T<K}$ and an asset-or-nothing option $S_T\mathbb{I}_{S_T<K}$ and value each separately.]

$$(K - S_T)_+ = (K - S_T)\,\mathbb{1}_{S_T<K}$$

$$= \underbrace{K\,\mathbb{1}_{S_T<K}}_{\text{option } A} - \underbrace{S_T\,\mathbb{1}_{S_T<K}}_{\text{option } B}$$

$$\frac{V_0^A}{1} = \mathbb{E}^{\mathbb{Q}}\left[\frac{V_T^A}{M_T}\right] \Rightarrow V_0^A$$

$$= \mathbb{E}^{\mathbb{Q}}\left[K\,\mathbb{1}_{S_T<K}\, e^{-rT}\right]$$

$$= K e^{-rT}\ \mathbb{Q}(S_T < K)$$

$$= K\ e^{-rT}\ \mathbb{Q}\left(Z^{\mathbb{Q}} < -\frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$\hookrightarrow \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$= K\ e^{-rT}\ \Phi(-d_-)$$

$$\frac{V_0^B}{S} = \mathbb{E}^{\mathbb{Q}^S}\left[\frac{V_T^B}{S_T}\right]$$

$$= \mathbb{E}^{\mathbb{Q}^S}\left[\frac{S_T \mathbb{1}_{S_T<K}}{S_T}\right] = \mathbb{E}^{\mathbb{Q}^S}\left[\mathbb{1}_{S_T<K}\right]$$

$$= \mathbb{Q}^S(S_T < K) = \mathbb{Q}^S\left(Z^{\mathbb{Q}^S} < -\frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$\overset{\mathbb{Q}^S}{\sim} N(0,1)$$

$$= \Phi(-d_+)$$

$$\Rightarrow \quad V = V_0^A - V_0^B$$

$$= K e^{-rT} \Phi(-d_-) - S\,\Phi(-d_+)$$