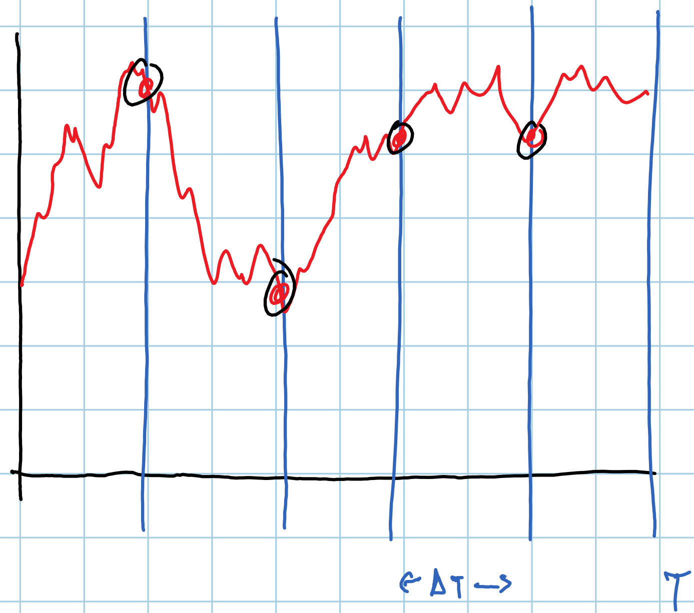
Move based hedging

Tuesday, November 27, 2012
2:08 PM

Δ_k units of S_{t_k}



$t_{k-1} \qquad\qquad t_k$

$\Delta_{k-1} \longrightarrow \Delta_{k-1} \quad \Delta_k$

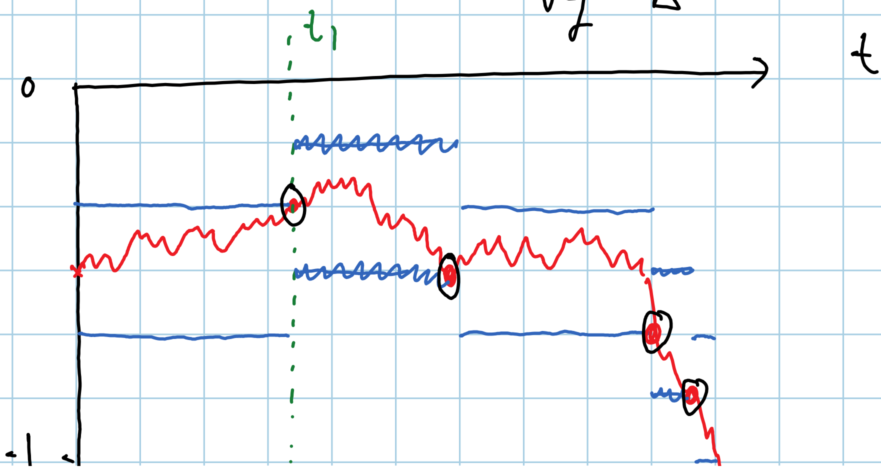
$S_{k-1} \longrightarrow S_k$

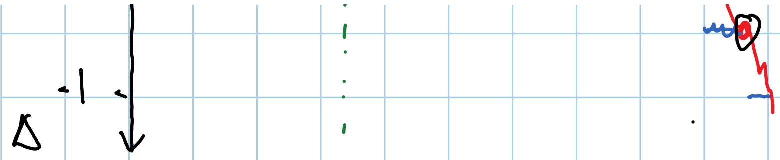
\$ in bank @ t_k^- $\qquad M_{k-1} e^{r \Delta t_k}$

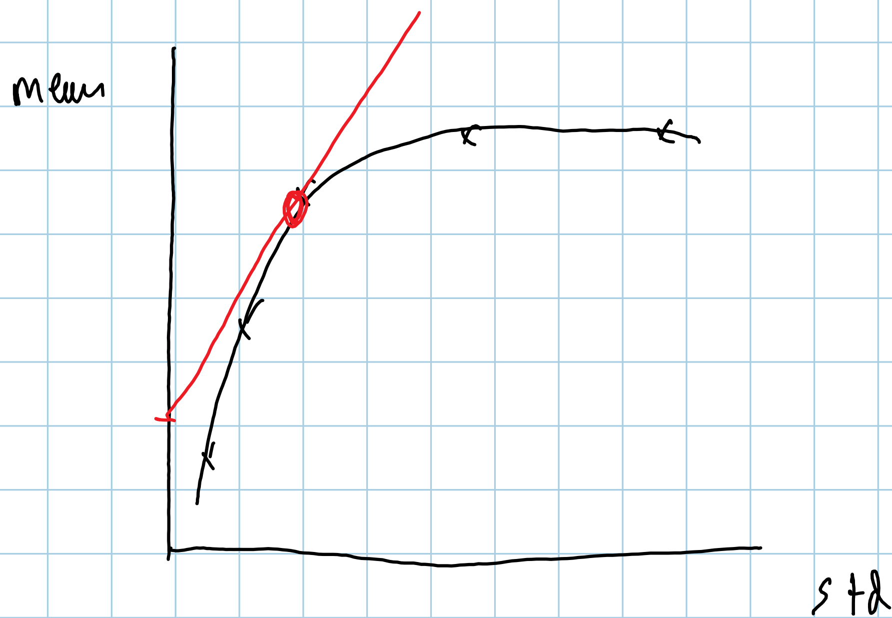
@ t_k $\qquad M_k = M_{k-1} e^{r \Delta t_k} - (\Delta_k - \Delta_{k-1}) S_k$

$-$ trans cost.

Move based : times at which you trade t_k are random ... but determined by Δ





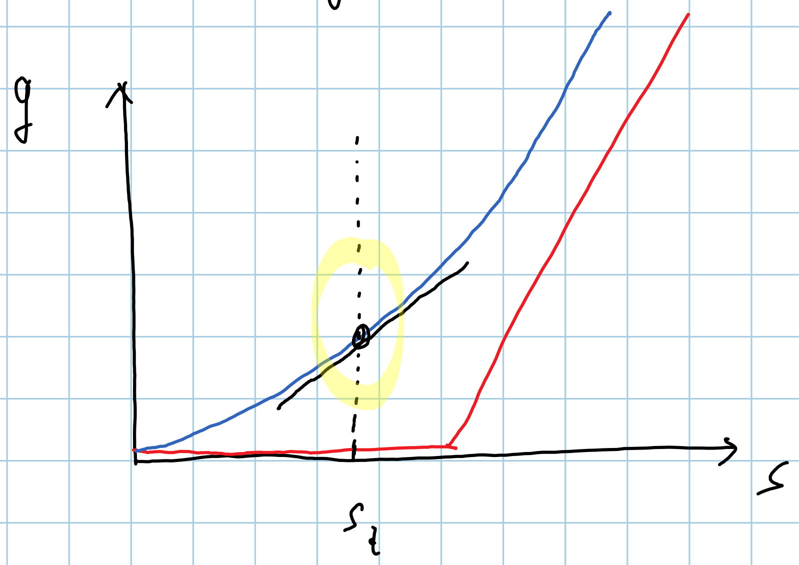


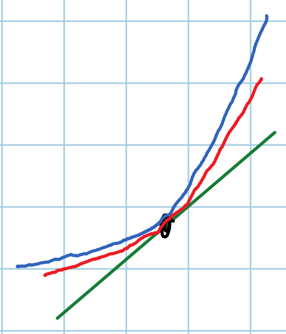
0.00625	-0.39	0.705
0.0125	-0.35	0.71
0.025	-0.34	0.75
0.05	-0.34	0.85
0.1	-0.32	1.29
0.2	-0.33	2.19

Delta-Gamma Hedging

Tuesday, November 27, 2012
3:12 PM

$$\Delta_t = \partial_S g_t$$





Δ

$$g(t,S) = g(t,S_t) + (S-S_t)\,\partial_S g(t,S_t) + \frac{1}{2}(S-S_t)^2\,\partial_{SS} g(t,S_t) + \dots$$

Γ

$\Gamma_t \triangleq \partial_{SS} g(t,S_t)$ option's gamma

Delta-Gamma hedging

α_t units of S_t

β_t units of M_t

γ_t units of another claim h_t

$$V(t,S) = \alpha_t S + \beta_t e^{rt} + \gamma_t h(t,S)$$

$$= \alpha_t S + \beta_t e^{rt} + \gamma_t \Big[h(t,S_t) + (S-S_t)\,\partial_S h(t,S_t) + \tfrac{1}{2}(S-S_t)^2\,\partial_{SS} h(t,S_t) \Big] + \cdots$$

$$\text{want} = g(t,S_t) + (S-S_t)\,\partial_S g(t,S_t) + \tfrac{1}{2}(S-S_t)^2\,\partial_{SS} g(t,S_t) + \cdots$$

$$\alpha_t + \gamma_t \underbrace{\partial_S h(t,S_t)}_{\Delta_t^h} = \underbrace{\partial_S g(t,S_t)}_{\Delta_t^g}$$

$$\gamma_t \underbrace{\partial_{SS} h(t,S_t)}_{\Gamma_t^h} = \underbrace{\partial_{SS} g(t,S_t)}_{\Gamma_t^g}$$

$$\boxed{\begin{aligned} \gamma_t &= \frac{\Gamma_t^g}{\Gamma_t^h} \\ \alpha_t &= \Delta_t^g - \frac{\Gamma_t^g}{\Gamma_t^h}\,\Delta_t^h \end{aligned}}$$

$$V = \alpha_t S + \beta_t M_t + \gamma_t \, h(t, S)$$

$$\Delta_t^V = \alpha_t + \gamma_t \underbrace{\partial_S h(t, S_t)}_{\Delta_t^h} = \Delta_t^g$$

$$\Gamma_t^V = \gamma_t \underbrace{\partial_{SS} h(t, S_t)}_{\Gamma_t^h} = \Gamma_t^g$$

<u>$t=0$</u> sold g $\rightarrow$ get g_0

need $\alpha_0 = \Delta_0^g - \dfrac{\Gamma_0^g}{\Gamma_0^h}\,\Delta_0^h$ of S

& $\gamma_0 = \dfrac{\Gamma_0^g}{\Gamma_0^h}$ of h

in bank $M_0 = g_0 - \alpha_0 S_0 - \gamma_0 h_0$

$t = t_1$:

$M_0 \longrightarrow M_0 e^{r\Delta t_1} \longrightarrow M_1 = M_0 e^{r\Delta t} - (\alpha_1 - \alpha_0) S_1 - (\gamma_1 - \gamma_0) h_1$

α_0 of S $(\alpha_0 S_0) \longrightarrow \alpha_0$ of S $(\alpha_0 S_1) \longrightarrow \alpha_1$ of S

γ_0 of h $(\gamma_0 h_0) \longrightarrow \gamma_0$ of h $(\gamma_0 h(t_1, S_1))$ $\hookrightarrow h_1$ $\longrightarrow \gamma_1$ of h (*)

$t = 0$ $\qquad$ $t = t_1^-$ $\qquad$ $t = t_1$

(*) in general is

$$M_k = M_{k-1} e^{r\Delta t_k} - (\alpha_k - \alpha_{k-1}) S_k - (\gamma_k - \gamma_{k-1}) h_k$$

$\vdots$

$t = t_N = T$

$$P\eta L = M_{N-1} e^{r\Delta t_N} + \alpha_{N-1} S_N + \gamma_{N-1} h_N - \underset{\underset{\varphi(S_T)}{\shortparallel}}{g_N}$$

Vega is sensitivity to vol:

$$\mathcal{V}_t = \partial_\sigma g(t, S_t)$$

$$\nu_t = \partial_\sigma g(t, S_t)$$

Options on Dividend Paying Assets

Tuesday, November 27, 2012
4:09 PM

options on asset with dividends

S_t pays dividends of $\delta S_t \, dt$ at t.

- $\frac{dS_t}{S_t} = \mu \, dt + \sigma \, dW_t$ is risky asset $\quad \alpha_t$
- B_t bank acct. $\quad dB_t = r B_t \, dt \quad \beta_t$
- value a claim on S, $g_t = g(t, S_t) \quad -1$

  $g \in C^{1,2}$

- $V_t = \alpha_t S_t + \beta_t B_t - g_t$

- $V_0 = 0$ start with nothing

- $dV_t = \alpha_t \, dS_t + \beta_t \, dB_t - dg_t + \alpha_t \delta S_t \, dt$

self-financing

$$= \alpha_t (\mu S_t \, dt + \sigma S_t \, dW_t)$$

$$+ \beta_t \, r B_t \, dt$$

$$- \Big[\Big(\partial_t g_t + \underbrace{\mu S_t \, \partial_S g_t + \tfrac{1}{2} \sigma^2 S_t^2 \, \partial_{SS} g_t}_{} \Big) dt$$

$$+ \sigma S_t \partial_s g_t \, dW_t]$$

$\mathcal{L} g_t$

$$+ \alpha_t \delta S_t \, dt$$

$$\boxed{\alpha_t = \partial_s g_t}$$

$$\Rightarrow \quad dV_t = \left\{ \alpha_t (\mu + \delta) S_t + \beta_t r B_t - (\partial_t + \mathcal{L}) g_t \right\} dt$$

since $dV_t = \{\cdot\} \, dt$ (i.e. it is predictable)

$\Rightarrow$ $\{\cdot\} = 0$ to avoid arbitrage

$\Rightarrow$ $dV_t = 0$ & since $V_0 = 0 \Rightarrow V_t = 0$

$$\Rightarrow \quad \alpha_t S_t + \beta_t B_t - g_t = 0$$

$$\Rightarrow \quad \boxed{\beta_t = B_t^{-1} (g_t - \alpha_t S_t)}$$

now sub α_t & β_t into $\{\cdot\} = 0$

$$\Rightarrow \quad \partial_s g_t (\cancel{\mu} + \delta) S_t + r (g_t - \partial_s g_t S_t)$$

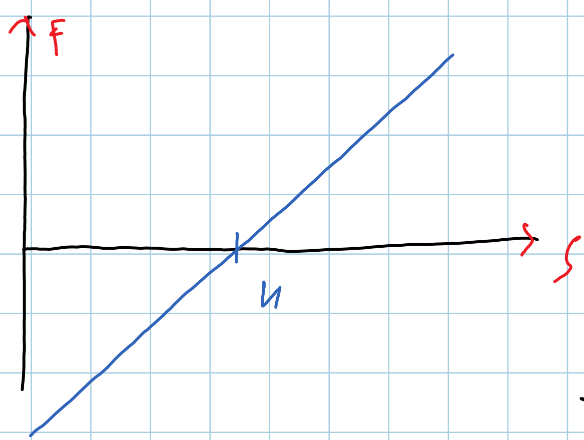
$$- (\partial_t g_t + \cancel{\mu S_t \partial_s g_t} + \tfrac{1}{2} \sigma^2 S_t^2 \partial_{ss} g_t) = 0$$

$$\Rightarrow \quad \partial_t g_t + (r - \delta) S_t \partial_s g_t + \tfrac{1}{2} \sigma^2 S_t^2 \partial_{ss} g_t = r g_t$$

$\forall$ payoffs of S_t $\Rightarrow$

$$\boxed{\begin{aligned} \left(\partial_t + (r-\delta) S \partial_S + \tfrac{1}{2}\sigma^2 S^2 \partial_{SS}\right) g(t,S) &= r\, g(t,S) \\ g(T,S) &= \varphi(S) \end{aligned}}$$

Forward price of an asset



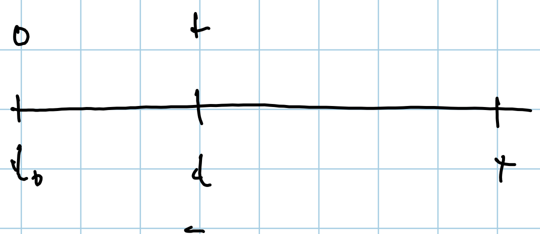
$$F_t(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)\right]$$

at signing $F_{t_0}(T) = 0$

sets K

$$\Rightarrow \quad K = \mathbb{E}^{\mathbb{Q}}[S_T \mid S_{t_0}]$$

$$= S_{t_0} e^{r(T-t_0)}$$



$F_t(T)$ is the strike in a forward contract if contract is signed on day t.

$$F_t(T) = \mathbb{E}^{\mathbb{Q}}[S_T \mid S_t] = S_t e^{r(T-t)}$$

$$F_t(T) = \mathbb{E}^{\mathbb{Q}}[S_T \mid S_t] = S_t e^{r(T-t)}$$

Futures contracts are like forward contracts except strike is adjusted each day and equals the <u>forward price</u> $F_t(T)$

called futures price.

$$\$2 \longrightarrow \$1.8 \longrightarrow \$1.9 \longrightarrow \$1.85$$

$$-0.2 \qquad +0.1 \qquad -0.05$$

costs nothing to enter / leave the contract.

But you pay the losses & reap the gains.

Options on Futures

Tuesday, November 27, 2012
4:43 PM

- $$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$$

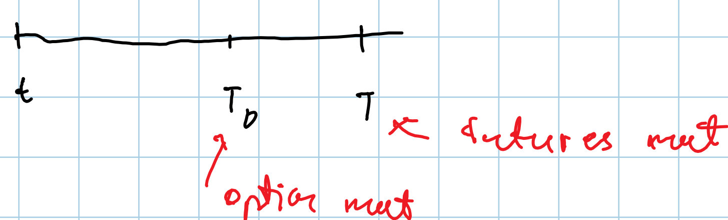
$$F_t(T) = S_t\, e^{r(T-t)}$$

$$dF_t(T) = dS_t\, e^{r(T-t)} - r\, S_t e^{r(T-t)}\, dt$$

$$\Rightarrow \frac{dF_t(T)}{F_t(T)} = (\mu - r)\, dt + \sigma\, dW_t \qquad \leftarrow (\alpha_t)$$

- $$dB_t = r\, B_t\, dt \qquad (\beta_t)$$

- claim $\quad g_t = g(t, F_t(T)) \qquad g_{T_0} = \varphi(F_{T_0}(T)) \qquad (-1)$



- $$V_t = \beta_t B_t - g_t$$

- $$dV_t = \beta_t\, dB_t - dg_t + \alpha_t\, dF_t(T)$$

self-financing

$$= \beta_t\, r\, B_t\, dt - \Big[\Big(\partial_t + (\mu - r) F_t\, \partial_F + \tfrac{1}{2}\sigma^2 F_t^2\, \partial_{FF}\Big) g_t$$

$$+ \sigma F_t \, \partial_F g_t \, dW_t \,]$$

$$+ \alpha_t \left((\mu - r) F_t \, dt + \sigma F_t \, dW_t \right)$$

$$\boxed{\alpha_t = \partial_F g(t, F_t)}$$

$\Rightarrow \quad dV_t = \{\cdot\} \, dt \quad \Rightarrow \quad \{\cdot\} = 0$ to avoid arbitrage b/c otherwise dV_t is predictable >0 (<0).

so since $V_0 = 0$, $dV_t = 0 \Rightarrow V_t = 0$

$$\Rightarrow \quad \beta_t B_t - g_t = 0 \quad \Rightarrow \quad \boxed{\beta_t = B_t^{-1} g_t}$$

$\{\cdot\} = 0$

$$\Rightarrow \quad r g_t - \left(\partial_t g_t + \cancel{(\mu - r) F_t \, \partial_F g_t} + \tfrac{1}{2} \sigma^2 \partial_{FF} g_t \right) + \cancel{\partial_F g_t \, (\mu - r) F_t} = 0$$

$$\Rightarrow \quad \partial_t g_t + \tfrac{1}{2} \sigma^2 F_t^2 \partial_{FF} g_t = r g_t$$

$\forall$ paths of F_t

$$\boxed{\begin{cases} \partial_t g(t,F) + \frac{1}{2}\sigma^2 F^2 \partial_{FF} g(t,F) = r g(t,F) \\ g(T_0, F) = \varphi(F) \end{cases}}$$