Feynman-Kac

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$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t \qquad (GBM)$$

$$\frac{dB_t}{B_t} = r\, dt$$

g_t is price of an option $g_t = g(t, S_t)$, $g \in C^{1,2}$

$$\boxed{\begin{cases} \left(\partial_t + r S \partial_S + \frac{1}{2}\sigma^2 S^2 \partial_{SS}\right) g(t,S) = r\, g(t,S) \\ g(T,S) = \varphi(S) \end{cases}}$$

$$g(t,S) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \varphi(S_T) \right]$$

$$S_T \stackrel{d}{=} S\, e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z}, \quad Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$\begin{cases} \partial_t h(t,x) + \frac{1}{2}\partial_{xx} h(t,x) = 0 \\ h(T,x) = H(x) \end{cases}$$

Heat equation

consider a B. motion X_t

$$F(t,x) \stackrel{\Delta}{=} \mathbb{E}\left[H(X_T) \mid X_t = x \right]$$

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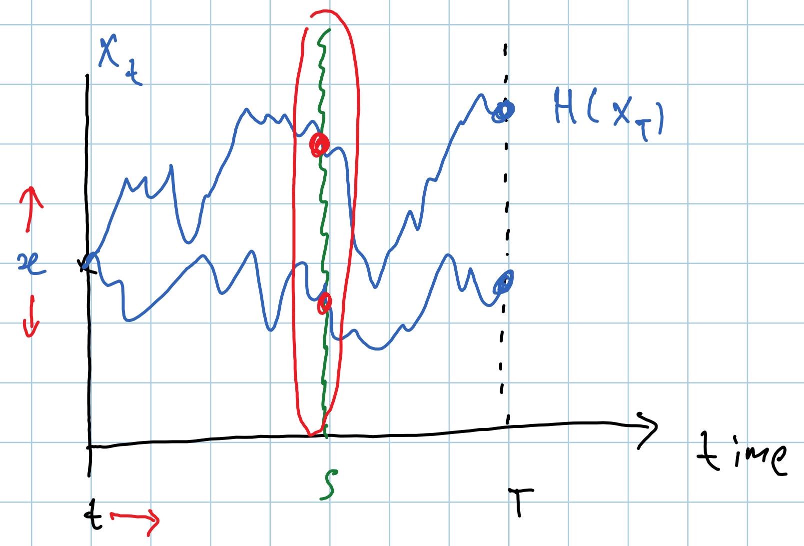

$$\eta_t = \mathbb{E}[\, \underbrace{f(s, X_s)} \mid X_t = x \,]$$

$f_s = f(s, X_s)$ is a r.v.

$$= \mathbb{E}[\, H(X_T) \mid X_s \,]$$

$$\eta_t = \mathbb{E}[\, f_s \mid X_t = x \,]$$

$$= \mathbb{E}[\, \underbrace{\mathbb{E}[\, H(X_T) \mid X_s \,]} \mid X_t = x \,]$$

$$= \mathbb{E}\Big[\overbrace{\mathbb{E}[\underbrace{H(X_T)}_{A} \mid X_s]}\mid X_t = x \Big]$$

$\uparrow$ B

↳ this is also a r.v.!

$$= \mathbb{E}[H(X_T) \mid X_t = x]$$ by iterated expectations

$(\mathbb{E}[\,\mathbb{E}[A \mid B] \mid C] = \mathbb{E}[A \mid C])$

$$= f(t, x)$$

$f_t = f(t, X_t)$ we have $\boxed{f_t = \mathbb{E}[f_s \mid X_t]}$ $t \le s \le T$

such processes are called martingales

take $s = t + \Delta t$

$$f_t = \mathbb{E}[f_{t+\Delta t} \mid X_t]$$

$$\Leftrightarrow \quad 0 = \mathbb{E}[\underbrace{f_{t+\Delta t} - f_t}_{\text{" } df_t \text{ "}} \mid X_t]$$

$$df_t = d f(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t + \frac{1}{2} \partial_{xx} f(t, X_t)\, dt$$

Ito's lemma

$$\Leftrightarrow \quad 0 = \left(\partial_t f(t, X_t) + \frac{1}{2} \partial_{xx} f(t, X_t) \right) \Delta t$$

$$\Rightarrow \quad \left(\partial_t + \frac{1}{2} \partial_{xx} \right) f(t, X_t) = 0$$

$\forall$ paths of X_t

$$\Rightarrow \boxed{\begin{cases} (\partial_t + \tfrac{1}{2}\partial_{xx}) f(t,x) = 0 \\ f(T,x) = H(x) \end{cases}}$$

Feynman-Kac Thm

recall $\quad f(t,x) = \mathbb{E}[\, H(X_T) \mid X_t = x]$

$$\Rightarrow f(T,x) = \mathbb{E}[\, H(X_T) \mid X_T = x] = H(x)$$

Hence $f(t,x)$ is a sol to the PDE!

example: $\quad H(x) = x$, then

$$f(t,x) = \mathbb{E}[\, X_T \mid X_t = x] = \mathbb{E}[\, X_t + (X_T - X_t) \mid X_t = x]$$

$$= \mathbb{E}[\, x + \underbrace{(X_T - X_t)}_{\sim N(0,\, T-t)} \mid X_t = x]$$

$$= x$$

$$\partial_t f = 0, \quad \partial_x f = 1, \quad \partial_{xx} f = 0$$

Clearly $(\partial_t + \tfrac{1}{2}\partial_{xx}) f = 0$ & $f(T,x) = x = H(x)$

so PDE is satisfied.

$$H(x) = x^2,$$

$$f(t,x) = \mathbb{E}[\,X_T^2 \mid X_t = x]$$

$$= \mathbb{E}[\,(x + \underbrace{(X_T - X_t)}_{\Delta X \sim N(0,\,T-t)})^2 \mid X_t = x]$$

$$= \mathbb{E}[\,x^2 + 2x\,\Delta X + \Delta X^2 \mid X_t = x]$$

$$= x^2 + 2x\,(0) + (T-t)$$

$$= x^2 + (T-t)$$

$$\partial_t f = -1, \quad \partial_x f = 2x, \quad \partial_{xx} f = 2$$

$$\Rightarrow (\partial_t + \tfrac{1}{2}\partial_{xx}) f = -1 + \tfrac{1}{2}(2) = 0$$

$$\& \quad f(T,x) = x^2 + 0 = H(x)$$

$\therefore$ satisfies PDE.

$$H(x) = e^{ax}$$

$$f(t,x) = \mathbb{E}[\,e^{aX_T} \mid X_t = x]$$

$$= \mathbb{E}[\,e^{a(x+\Delta X)} \mid X_t = x]$$

$$= e^{ax}\,\mathbb{E}[\,e^{a\Delta X} \mid X_t = x]$$

$$= e^{ax} \, \mathbb{E}[\, e^{a\Delta X} \mid X_t = x]$$

$$= e^{ax} \, e^{\frac{1}{2}a^2(T-t)}$$

$$\Rightarrow f(t,x) = e^{ax + \frac{1}{2}a^2(T-t)}$$

$$\partial_t f = -\frac{1}{2}a^2 f, \quad \partial_x = a f, \quad \partial_{xx} = a^2 f$$

$$(\partial_t + \tfrac{1}{2}\partial_{xx}) f = -\frac{1}{2}a^2 f + \frac{1}{2}(a^2 f) = 0$$

$$f(T,x) = e^{ax+0} = H(x)$$

$\therefore$ PDE satisfied!

F-K for B-S

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<u>Feynman-Kac</u> Thm:

a solution to the PDE

$$\begin{cases} \left[\partial_t + a(t,x)\,\partial_x + \frac{1}{2} b^2(t,x)\,\partial_{xx}\right] h(t,x) = c(t,x)\,h(t,x) \\ h(T,x) = H(x) \end{cases}$$

is given by:

$$h(t,x) = \mathbb{E}\left[e^{-\int_t^T c(u,X_u)\,du} H(X_T) \,\middle|\, X_t = x \right]$$

where the stochastic process X_t satisfies the SDE:

$$dX_t = a(t,X_t)\,dt + b(t,X_t)\,dW_t$$

Black-Scholes PDE:

$$\begin{cases} \left(\partial_t + r\,S\,\partial_S + \frac{1}{2}\sigma^2 S^2 \partial_{SS}\right) g(t,S) = r\,g(t,S) \\ g(T,S) = \varphi(S) \end{cases}$$

has solution:

$$\boxed{\begin{aligned} & g(t,S) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \cdot \varphi(S_T) \,\middle|\, S_t = S \right] \\ & \& \quad dS_t = r\,S_t\,dt + \sigma\,S_t\,d\widehat{W}_t . \end{aligned}}$$

we started with $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$

we started with $dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$

e.g. $\varphi(S) = S$.

$$g(t,S) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} S_T \mid S_t = S \right]$$

$$dS_t = r S_t \, dt + \sigma S_t \, d\hat{W}_t$$

$$\Rightarrow \quad d \ln S_t = \left(r - \tfrac{1}{2}\sigma^2\right) dt + \sigma \, d\hat{W}_t$$

$$\Rightarrow \quad \ln S_T - \ln S_t = \left(r - \tfrac{1}{2}\sigma^2\right)(T-t) + \sigma\left(\hat{W}_T - \hat{W}_t\right)$$

$$\Rightarrow \quad S_T = S_t \, e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(\hat{W}_T - \hat{W}_t\right)}$$

$$\overset{d}{=} S_t \, e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\, Z}, \quad Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

$$g(t,S) = e^{-r(T-t)} \, \mathbb{E}\left[S_t \, e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\, Z} \mid S_t = S \right]$$

$$= S \, e^{-\frac{1}{2}\sigma^2 (T-t)} \, \underbrace{\mathbb{E}\left[e^{\sigma\sqrt{T-t}\, Z} \right]}_{\hookrightarrow \; e^{\frac{1}{2}\sigma^2 (T-t)}}$$

$$= S.$$

$$\partial_t g = 0, \quad \partial_S g = 1, \quad \partial_{SS} g = 0$$

$$\overbrace{\underset{\substack{\| \\ 0}}{\partial_t g} + r S \underset{\substack{\| \\ 1}}{\partial_S g} + \tfrac{1}{2}\sigma^2 S^2 \underset{\substack{\| \\ 0}}{\partial_{SS} g}} = r S = \boxed{r \, g}$$

$$g(T, S) = S = \Phi(S)$$

$\Rightarrow$ Black-Scholes PDE is satisfied!

$$\Phi = S^3 \qquad g = \mathbb{E}[\;\cdot\cdot\;] \longrightarrow PDE$$

Delta-Gamma Sketches

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Delta: $\Delta(t,S) = \partial_S g(t,S)$

Gamma: $\Gamma(t,S) = \partial_{SS}\, g(t,S)$

call option:

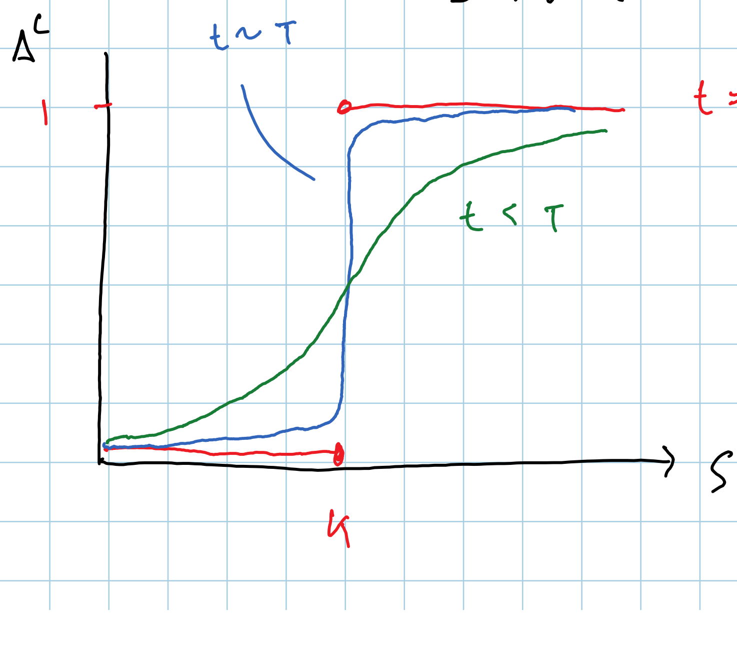
$$g(t,S) = S\,\Phi(d_+) - K e^{-r(T-t)}\,\Phi(d_-)$$

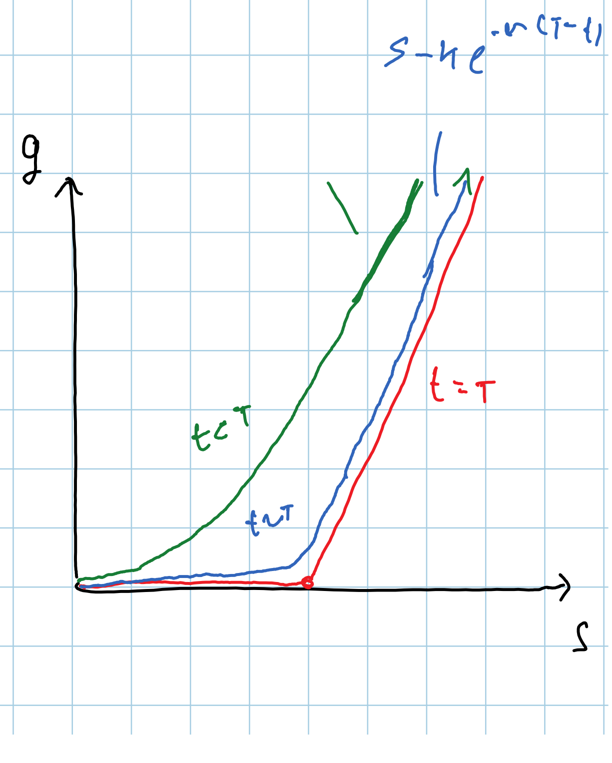
$$d_\pm = \frac{\ln(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

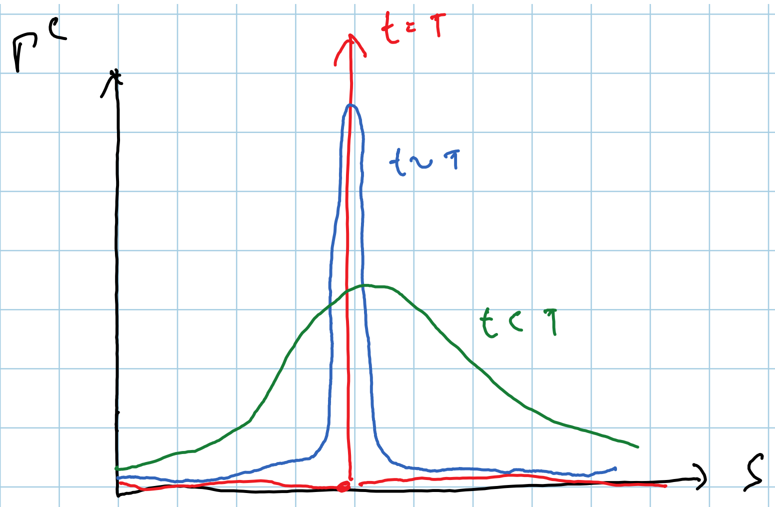
$$\Delta^c = \partial_S g = \Phi(d_+)$$

$$\Gamma^c = \partial_{SS} g = \Phi'(d_+)\cdot \partial_S d_+ \qquad \Phi'(x) = \phi(x)$$

$$= \frac{\phi(d_+)}{S\,\sigma\sqrt{T-t}}$$







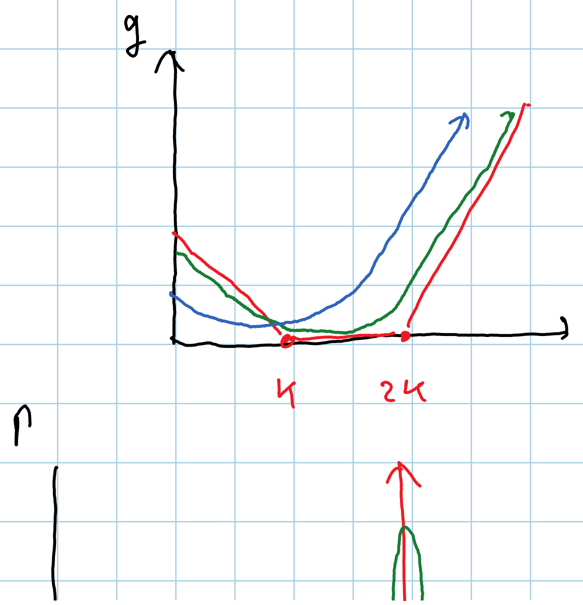
$$C - P = S - Ke^{-rt}$$ put-call parity

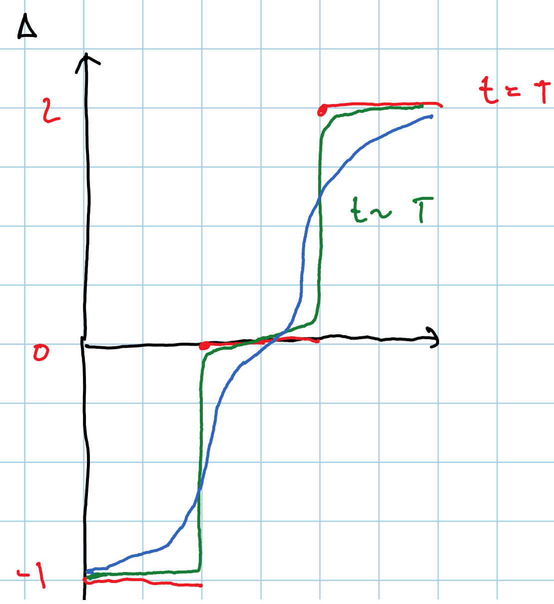
$\Rightarrow \quad \Delta^c - \Delta^p = 1$

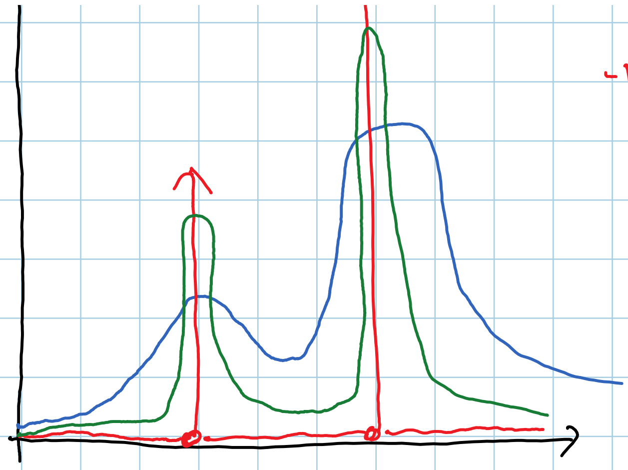
$\Rightarrow \quad \Delta^p = \Delta^c - 1$

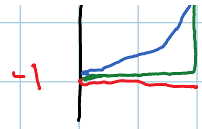
$\Rightarrow \quad \Gamma^p = \Gamma^c$

1 long put @ K   2 long calls @ 2K





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