Alternate CRR model

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$$A \begin{cases} \xrightarrow{p} A e^{\sigma\sqrt{\Delta t}} \\ A e^{-\sigma\sqrt{\Delta t}} \end{cases}$$

$$p = \frac{1}{2}\left[1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t} \right]$$

$$\mathbb{E}^{\mathbb{P}}[A_1] \sim e^{\mu \Delta t} A$$

$$\mathbb{V}^{\mathbb{P}}\left[\ln\left(\frac{A_1}{A}\right)\right] \sim \sigma^2 \Delta t$$

$$q = \cdots + \cdots \sqrt{\Delta t} + \cdots$$

$$p = \frac{1}{2}$$

$$A \begin{cases} \xrightarrow{p} A e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} \\ A e^{(\mu - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}} \end{cases}$$

$$1 \begin{cases} e^{r\Delta t} \\ e^{r\Delta t} \end{cases}$$

$$\mathbb{E}^{\mathbb{P}}[A_1] = A e^{(\mu - \frac{1}{2}\sigma^2)\Delta t}\left[\frac{1}{2} e^{\sigma\sqrt{\Delta t}} + \frac{1}{2} e^{-\sigma\sqrt{\Delta t}}\right]$$

$$\frac{1}{2}\left[\left(1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + \cdots\right) + \left(1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2 \Delta t + \cdots\right)\right]$$

$$= 1 + \frac{1}{2}\sigma^2 \Delta t + \cdots$$

$$= e^{\frac{1}{2}\sigma^2 \Delta t} + \cdots$$

$$= A e^{\mu \Delta t} + \cdots$$

$$\mathbb{V}^{\mathbb{P}}[\ln A_1] = \sigma^2 \Delta t$$

$$\mathbb{V}^{\mathbb{P}}[\ln A_1] = \sigma^2 \Delta t$$

$$A \overset{q}{\begin{cases} A e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} \\ A e^{(\mu - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}} \end{cases}} \qquad 1 \begin{cases} e^{r\Delta t} \\ e^{r\Delta t} \end{cases}$$

$$A = e^{-r\Delta t}\,\mathbb{E}^{\mathbb{Q}}[A_1]$$

$$A = e^{-r\Delta t}\left[q\, A\, e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} + (1-q)\, e^{(\mu - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}} \right]$$

$$\Rightarrow \quad q = \frac{e^{r\Delta t} - e^{(\mu - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}}{e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} - e^{(\mu - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}}$$

$$= \frac{e^{(\underbrace{r - (\mu - \frac{1}{2}\sigma^2)}_{\hat{r}})\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

$$\sim \frac{(\cancel{1} + \hat{r}\Delta t) - (\cancel{1} - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + \cdots}{(1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) - (1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + \cdots}$$

$$= \frac{\sigma\sqrt{\Delta t} + (\hat{r} - \frac{1}{2}\sigma^2)\Delta t + \cdots}{2\sigma\sqrt{\Delta t} + \cdots}$$

$$= \frac{1}{2}\left[1 + \frac{\hat{r} - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t} \right] + \cdots$$

$$= \frac{1}{2}\left[1 + \frac{r-\mu}{\sigma}\sqrt{\Delta t}\right] + \cdots$$

$$\left(q_{CRR} = \frac{1}{2}\left[1 + \frac{r-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right] + \cdots \right)$$

show that $\cancel{\mathbb{E}^{\mathbb{Q}}[A_1] = e^{r\Delta t} A}$

$$\mathbb{V}^{\mathbb{Q}}\left[\ln\left(\frac{A_1}{A}\right)\right] = \sigma^2 \Delta t + \cdots$$

$$A \begin{array}{l} \nearrow^{p} \; A\, e^{(\alpha - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} \\ \searrow \; A\, e^{(\alpha - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}} \end{array}$$

determine $\mathbb{P}$ s.t. $\mathbb{E}^{\mathbb{P}}[A_1] = e^{\mu\Delta t} + \cdots$

(when $\alpha = \mu$, $p = \frac{1}{2}$) $p = \text{?} + \text{?}\sqrt{\Delta t} + \cdots$

$$\overset{?}{=} \frac{1}{2}\left(1 + \frac{(\mu-\alpha)}{\sigma}\sqrt{\Delta t}\right) + \cdots \; ?$$

Forward Starting Option

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$$A_T \overset{d}{=} A_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z}, \qquad Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

Euro call option pays $\varphi = (A_T - K)_+$ @ T

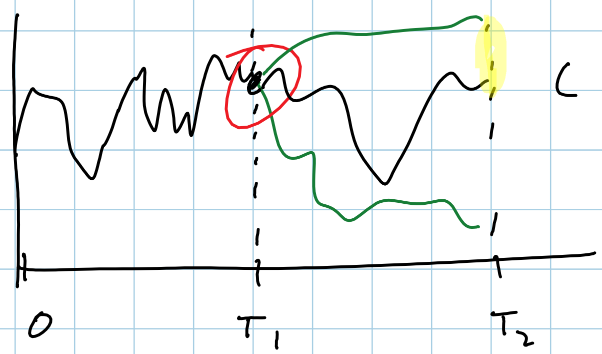
$$V_0 = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[(A_T - K)_+\right]$$

$$= \ldots$$

$$= A_0\, \Phi(d_+) - K e^{-rT}\, \Phi(d_-)$$

$$d_\pm = \frac{\ln(A_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Forward starting call pays $\varphi = (S_{T_2} - \alpha S_{T_1})_+$



$(S_{T_2} - \alpha S_{T_1})_+$

$(\cdot)_+ \overset{\Delta}{=} \max(\cdot, 0)$

$$V_0 = e^{-rT_2}\, \mathbb{E}^{\mathbb{Q}}\left[(S_{T_2} - \alpha S_{T_1})_+\right]$$

$$S_{T_1} \overset{d}{=} S_0\, e^{(r-\frac{1}{2}\sigma^2)T_1 + \sigma\sqrt{T_1}\, Z_1}$$

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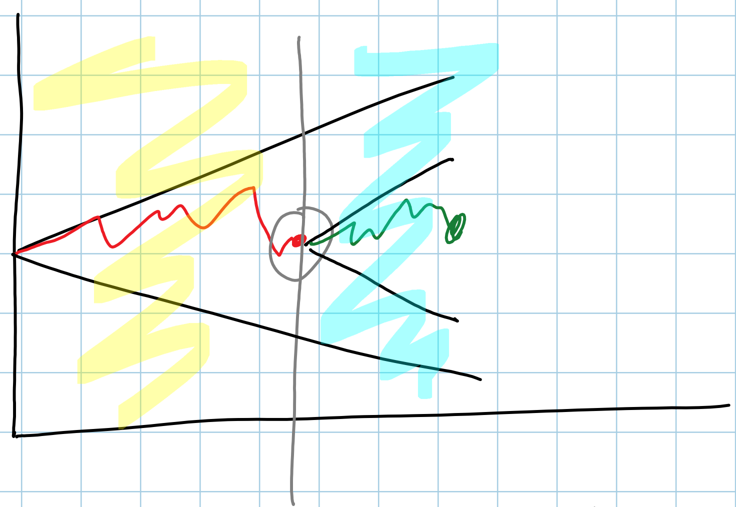
$$Z_1 \underset{Q}{\sim} N(0,1)$$

$$S_{T_2} \overset{d}{=} S_0\, e^{(r-\frac{1}{2}\sigma^2)T_2 + \sigma\sqrt{T_2}\, Z_2}$$

$$Z_2 \underset{Q}{\sim} N(0,1)$$



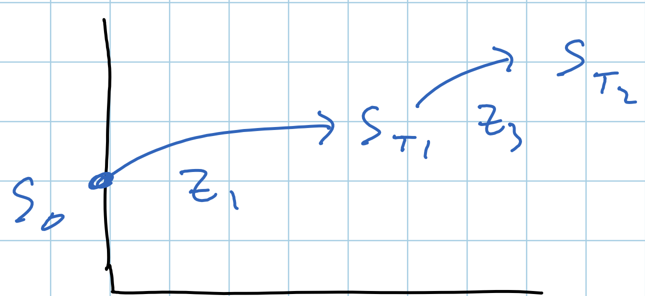
$$S_{T_2} \overset{d}{=} S_{T_1} \cdot e^{(r-\frac{1}{2}\sigma^2)(T_2-T_1) + \sigma\sqrt{T_2-T_1}\, Z_3}$$

$$Z_3 \underset{Q}{\sim} N(0,1)$$

$$S_{T_1} \overset{d}{=} S_0\, e^{(r-\frac{1}{2}\sigma^2)T_1 + \sigma\sqrt{T_1}\, Z_1}$$

$$Z_1 \underset{Q}{\sim} N(0,1)$$

Z_3 & Z_1 are independent.



$$V_0 = e^{-rT_2}\,\mathbb{E}^{\mathbb{Q}}\left[(S_{T_2} - \alpha S_{T_1})_+ \right]$$

$$= e^{-rT_2}\,\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[(S_{T_2} - \alpha S_{T_1})_+ \mid S_{T_1} \right] \right]$$

$$= e^{-rT_1}\,\mathbb{E}^{\mathbb{Q}}\Big[\underbrace{e^{-r(T_2-T_1)}\,\mathbb{E}^{\mathbb{Q}}\left[(S_{T_2} - \alpha S_{T_1})_+ \mid S_{T_1} \right]}_{\text{Black-Scholes with strike } \alpha S_{T_1}} \Big]$$

$$S_{T_1}\,\Phi(d_+) - \alpha S_{T_1} e^{-r(T_2-T_1)}\,\Phi(d_-)$$

$$d_\pm = \frac{\ln\left(S_{T_1}/(\alpha S_{T_1})\right) + \left(r \pm \frac{1}{2}\sigma^2\right)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}$$

= const. (not a r.v.)

$$= e^{-rT_1}\,\mathbb{E}^{\mathbb{Q}}[S_{T_1}]\,\eta \qquad , \eta = \Phi(d_+) - \alpha e^{-r(T_2-T_1)}\Phi(d_-)$$

$$= \eta\, S_0$$

MC simulation

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10:18 AM

Monte Carlo Simulations.

strong law of large numbers...

r.v. X $\mathbb{E}[|X|] < +\infty$, take independent

draws of X: $X^{(1)}, X^{(2)}, \dots, X^{(M)}$

$$\lim_{M \to +\infty} \frac{X^{(1)} + X^{(2)} + \dots X^{(M)}}{M} = \mathbb{E}[X].$$

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can estimate $m_1 = \mathbb{E}[X]$ by taking a finite sample

$$\hat{m}_1 = \frac{1}{M} \sum_{m=1}^{M} X^{(m)}$$ → sample mean

$$\hat{\sigma}_{m_1} = \frac{\left(\frac{1}{M-1} \sum_{m=1}^{M} (X^{(m)} - \hat{m}_1)^2 \right)^{\frac{1}{2}}}{M^{1/2}}$$ → sample variance

MC Simulation cont'd

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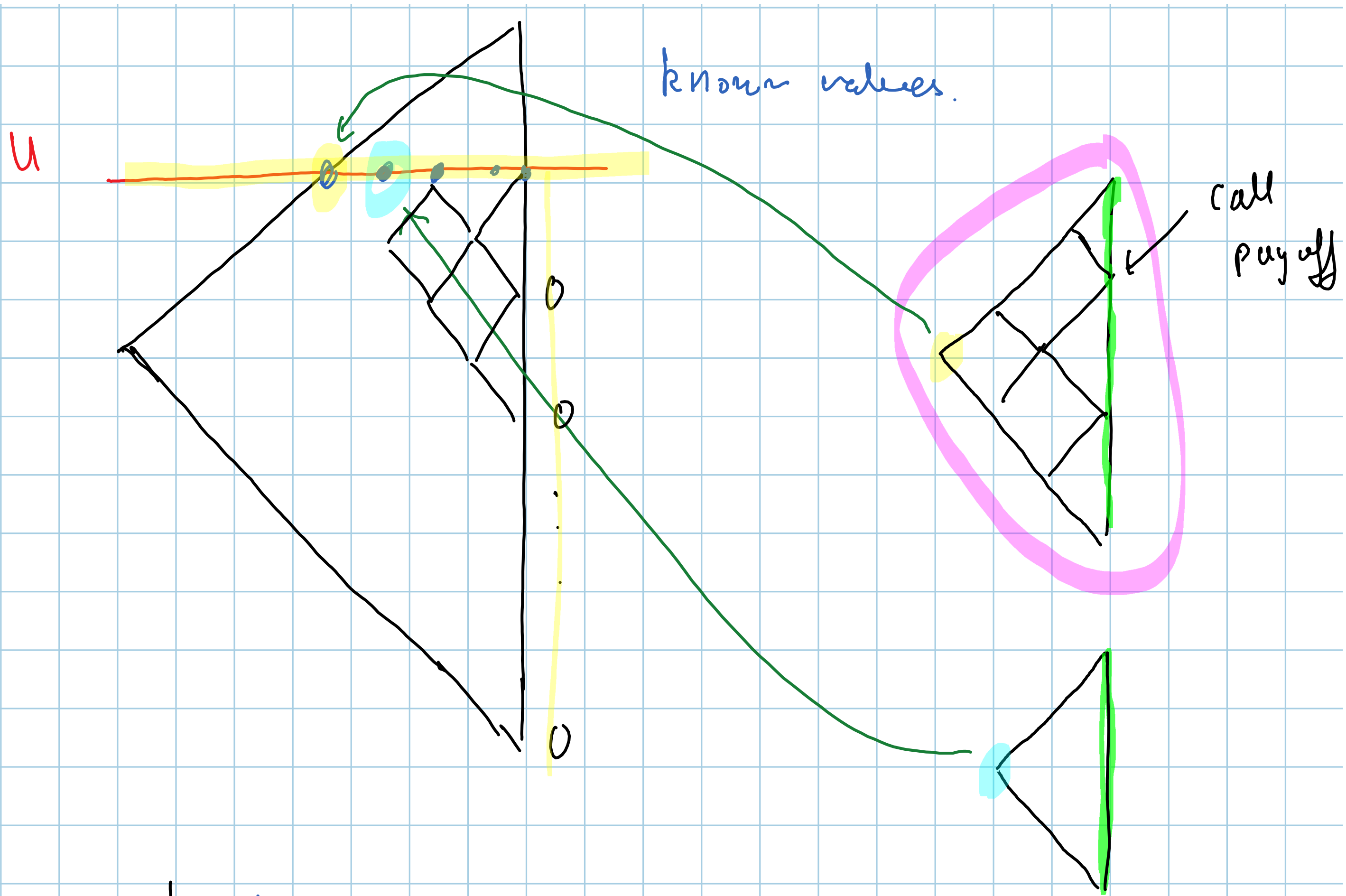
$S_0 \to S_{T_1} \longrightarrow S_{T_2}$

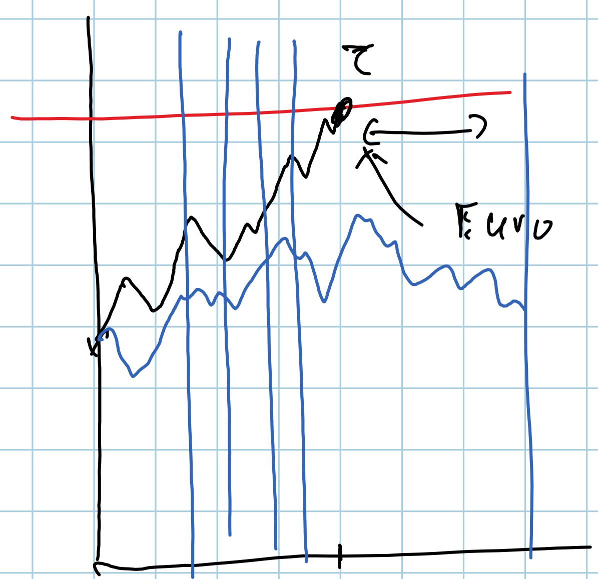
$\mathbb{C}[S_{T_2}, S_{T_1}] = \dots$

$$\rho = \frac{\mathbb{C}[S_{T_2}, S_{T_1}]}{\left(\mathbb{V}[S_{T_1}]\,\mathbb{V}[S_{T_2}]\right)^{1/2}}$$

Barrier options:

Knock In $\longrightarrow$ up and in call option





$t_n - t_{n-1}$

$$S_{t_n} \overset{d}{=} S_{t_{n-1}} e^{(r-\frac{1}{2}\sigma^2)\Delta t_n + \sigma\sqrt{\Delta t_n}\, Z_n}$$

$Z_1, Z_2, \ldots$ iid $\underset{\mathbb{Q}}{\sim} N(0,1)$