Stochastic Integral

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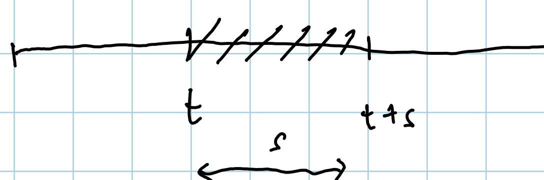
Brownian motion

- $W_0 = 0$
- $W_t \sim \mathcal{N}(0, t)$
- W_t has stationary increments

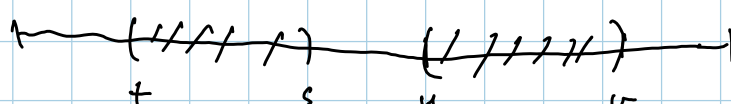
$$W_{t+s} - W_t \sim F(s)$$
$$\mathcal{N}(0, s)$$



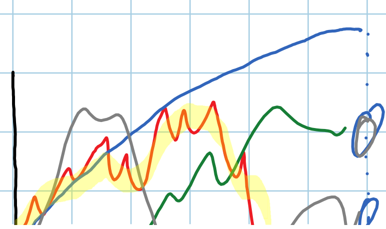
- W_t has independent increments

$$0 \le t < s \le u < v$$

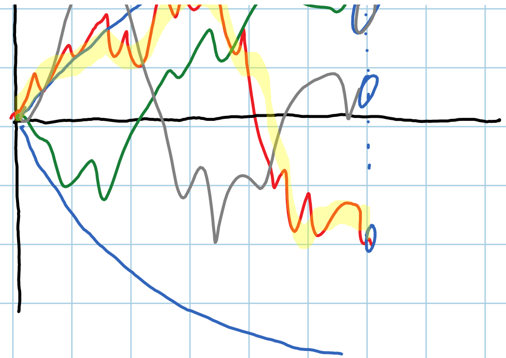
$$(W_s - W_t) \perp (W_v - W_u)$$



- W_t has continuous paths



$$TV(W)_t = \lim_{\|\Pi\| \downarrow 0} \sum_k \left| W_{t_k} - W_{t_{k-1}} \right|$$



$\|\Pi\|\downarrow 0$

$= +\infty$ (nowhere differentiable)

$$[W,W]_t = \lim_{\|\Pi\|\downarrow 0} \sum_k \left(W_{t_k} - W_{t_{k-1}}\right)^2$$

$$= t \quad \text{a.s.}$$

$$W_t \sim \mathcal{N}(0,t)$$

$$I_t = \int_0^t W_s \, dW_s \triangleq \lim_{\|\Pi\|\downarrow 0} \sum_k W_{t_{k-1}} \Delta W_{t_k}$$

Itô integral

$t_0 \quad t_1 \quad t_2 \quad t_3 \quad \dots \quad t_n = t$

$$I_t = \tfrac{1}{2}\left(W_t^2 - W_0^2\right) - \tfrac{1}{2} t \qquad \text{guess.}$$

$$\mathcal{E} = \left(\sum_k W_{t_{k-1}} \Delta W_{t_k}\right) - \left[\frac{1}{2} W_t^2 - \frac{1}{2} t\right]$$

want to show that $\mathcal{E} \xrightarrow[\|\Pi\|\downarrow 0]{} 0$ a.s.

- $\mathbb{E}[\mathcal{E}] \xrightarrow[\|\Pi\|\downarrow 0]{} 0$
- $\mathbb{V}[\mathcal{E}] \xrightarrow[\|\Pi\|\downarrow 0]{} 0$

$$\mathbb{E}[\varepsilon] = \sum_k \mathbb{E}[W_{t_{k-1}} \Delta W_{t_k}] - \left(\frac{1}{2}\mathbb{E}[W_t^2] - \frac{1}{2}t\right)$$

$$\mathbb{E}[W_{t_{k-1}}]\,\mathbb{E}[\Delta W_{t_k}] \qquad \mathbb{E}[W_{t_{k-1}}] \to 0, \quad \mathbb{E}[\Delta W_{t_k}] \to 0, \quad \mathbb{E}[W_t^2] \to t$$

$$= 0$$

$$\varepsilon = \sum_k W_{t_{k-1}} \Delta W_{t_k} - \frac{1}{2}\sum_k \underbrace{\left(W_{t_k}^2 - W_{t_{k-1}}^2 - \Delta t_k\right)}_{\Delta X_{t_k}}$$

$$(W_t^2 - t) = X_t$$

$$\Rightarrow \varepsilon = \sum_k \Big(\underbrace{W_{t_{k-1}}(W_{t_k} - W_{t_{k-1}}) - \frac{1}{2}W_{t_k}^2 + \frac{1}{2}W_{t_{k-1}}^2}_{= -\frac{1}{2}(W_{t_k} - W_{t_{k-1}})^2} + \frac{1}{2}\Delta t_k\Big)$$

$$= \frac{1}{2}\sum_k \left(\Delta t_k - \Delta W_{t_k}^2\right) \quad \left(= \frac{1}{2}\left(t - [W,W]_t^{\pi}\right)\right)$$

$$[W,W]_t^{\pi} \xrightarrow[\|\pi\| \downarrow 0]{} t \text{ a.s.}$$

$$\mathbb{V}[\varepsilon] = \frac{1}{4}\mathbb{V}\left[\sum_k \Delta W_{t_k}^2\right]$$

$$= \frac{1}{4} \sum_k \mathbb{V}[\Delta W_{t_k}^2]$$

$\Delta W_{t_k} \sim N(0; \Delta t_k)$

$$= \frac{1}{4} \sum_k \mathbb{V}[(\sqrt{\Delta t_k} \cdot Z_k)^2]$$

$Z_1, Z_2 \ldots \sim N(0,1)$

$$= \frac{1}{4} \sum_k \Delta t_k^2 \, \mathbb{V}[Z^2]$$

$\Delta t_k^2 = \Delta t_k \Delta t_k \leq \Delta t_k \|\Pi\|$

$$\leq \frac{1}{4} \|\Pi\| \underbrace{\sum_k \Delta t_k}_{t} \mathbb{V}[Z^2]$$

$$\xrightarrow[\|\Pi\| \downarrow 0]{} 0$$

- $\mathbb{E}[\mathcal{E}] = 0$. $\circ$ $\mathbb{V}[\mathcal{E}] \xrightarrow[\|\Pi\| \downarrow 0]{} 0$

$$\text{LLN} \Rightarrow \quad \mathcal{E} \xrightarrow[\|\Pi\| \downarrow 0]{} 0 \quad \text{a.s.}$$

$$\therefore \quad \boxed{\int_0^t W_s \, dW_s = \frac{1}{2}(W_t^2 - t)} \quad \text{a.s.}$$

Ito correction

convexity correction

Some Computations

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$$\mathbb{V}[W_t^2] = \mathbb{E}[W_t^4] - (\mathbb{E}[W_t^2])^2$$

$$= \mathbb{E}[(\sqrt{t}\,Z)^4] - t^2$$

$$= t^2(\mathbb{E}[Z^4] - 1)$$

$$\mathbb{E}[Z^4] = \int_{-\infty}^{\infty} z^4 \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\, dz$$

$$\mathbb{E}[e^{aZ}] = e^{\frac{1}{2}a^2} = h(a)$$

$$\partial_a^4 h\Big|_{a=0} = \mathbb{E}[Z^4 e^{aZ}]\Big|_{a=0} = \mathbb{E}[Z^4]$$

$$h' = a\, e^{\frac{1}{2}a^2}$$

$$h'' = e^{\frac{1}{2}a^2} + a^2 e^{\frac{1}{2}a^2}$$

$$= (a^2+1)\, e^{\frac{1}{2}a^2}$$

$$h''' = (2a)\, e^{\frac{1}{2}a^2} + (a^2+1)\, a\, e^{\frac{1}{2}a^2}$$

$$= (a^3 + 3a)\, e^{\frac{1}{2}a^2}$$

$$h'''' = (3a^2+3)\, e^{\frac{1}{2}a^2} + (a^3+3a)\, a\, e^{\frac{1}{2}a^2}$$

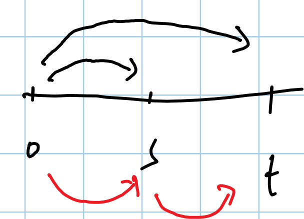
$$= (a^4 + 6a^2 + 3)\, e^{\frac{1}{2}a^2}$$

$$= (a^4 + 6a^2 + 3)\, e^{\frac{1}{2}a^2}$$

$$h''''(0) = 3$$

$$\therefore \ \mathbb{V}[W_t^2] = t^2(\mathbb{E}[Z^4]-1) = 2t^2.$$

- $\mathbb{E}[W_t W_s]$ $\qquad 0<s<t$



$$= \mathbb{E}[((W_t - W_s) + W_s)\, W_s]$$

$$= \mathbb{E}[(W_t-W_s)W_s + W_s^2]$$

$$= s$$

¿ (W_t, W_s) has what distribution? $(0<s<t)$

$$\begin{pmatrix} (W_t - W_s) + W_s \\ W_s \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} t & s \\ s & s \end{pmatrix} \right)$$

- $\mathbb{V}[W_t W_s] = \mathbb{E}[W_s^2 W_t^2] - (\underbrace{\mathbb{E}[W_t W_s]}_{s})^2$

$$\mathbb{E}[W_s^2(W_s + (W_t - W_s))^2]$$

$$= \mathbb{E}[W_s^2(W_s^2 + 2W_s(W_t-W_s) + (W_t-W_s)^2)]$$

$$= \mathbb{E}[W_s^4] + 2\mathbb{E}[W_s^3(W_t-W_s)] + \mathbb{E}[W_s^2(W_t-W_s)^2]$$

$= \mathbb{E}[W_s^4] + 2\mathbb{E}[W_s^3(W_t - W_s)] + \mathbb{E}[W_s^2(W_t - W_s)^2]$

$3s^2 \quad + \quad 0 \qquad\qquad + \quad s \cdot (t-s)$

$= 2s^2 + st$

$\therefore \mathbb{V}[W_t^2 W_s^2] = 2s^2 + st - s^2$

$= s(t+s)$

$$X_t = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$$

$$\mathbb{E}[X_t] = \mathbb{E}[e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}]$$

$Z \sim N(0,1)$

$$= e^{-\frac{1}{2}\sigma^2 t}\, \mathbb{E}[e^{\sigma W_t}] \quad \mathbb{E}[e^{\sigma\sqrt{t} Z}]$$

$$= e^{-\frac{1}{2}\sigma^2 t}\, e^{\frac{1}{2}\sigma^2 t}$$

$$= 1$$

$\Uparrow \quad dX_t = \sigma X_t \, dW_t \leftarrow$

$$\mathbb{E}[X_t^2] = \mathbb{E}[e^{-\sigma^2 t + 2\sigma W_t}]$$

$$= e^{-\sigma^2 t}\, \mathbb{E}[e^{2\sigma\sqrt{t} Z}]$$

$$= e^{-\sigma^2 t} \cdot e^{\frac{1}{2}\cdot 4\sigma^2 t}$$

$$= e^{\sigma^2 t}$$

Ito's Lemma

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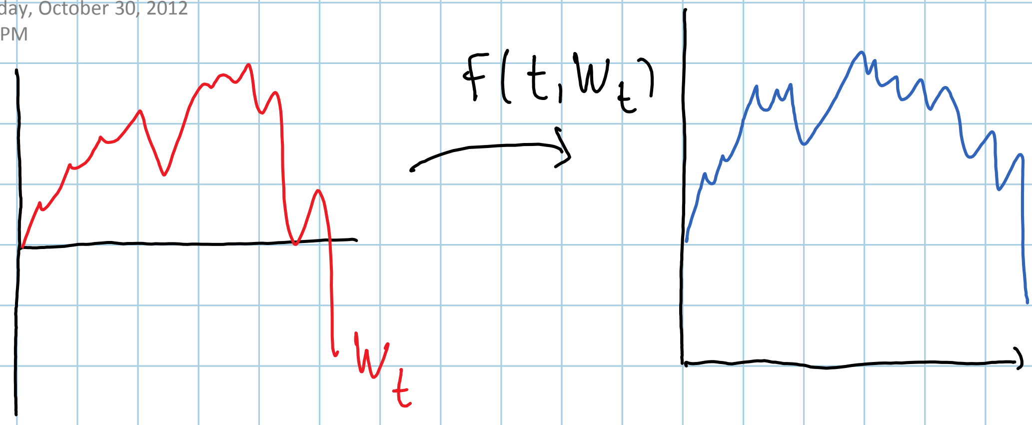

$X_t = f(t, W_t)$

e.g. $f(t, \omega) = e^{-\frac{1}{2}\sigma^2 t + \sigma \omega}$

$$X_{t+\Delta t} - X_t \overset{?}{\doteq} (\quad)\,\Delta t + (\quad)\,(W_{t+\Delta t} - W_t)$$

Yes, if $f \in C^{1,2}$ (1 diff in t, 2 diff. in ω)

then,

$$dX_t = \left[\partial_t f(t, W_t) + \frac{1}{2}\partial_{\omega\omega} f(t, W_t)\right] dt$$

$$+ \partial_\omega f(t, W_t)\, dW_t$$

SDE

Ito's lemma for Brownian motions.

$$\left(X_t - X_0 = \int_0^t \left(\partial_t f(s, W_s) + \frac{1}{2}\partial_{\omega\omega} f(s, W_s)\right) ds \right.$$

$$\left. + \int_0^t \partial_\omega f(s, W_s)\, dW_s \right)$$

$$+\underbrace{\int_0^t \partial_w f(s, W_s)\, dW_s}_{\text{Ito-integral}}$$

<u>Ito-integral</u>:

$$I_t = \int_0^t g(s, W_s)\, dW_s$$

$$\triangleq \lim_{\|\Pi\| \downarrow 0} \sum_k g(t_{k-1}, W_{t_{k-1}})\, \Delta W_{t_k}$$

Use Ito lemma to compute $\int_0^t W_s\, dW_s = I_t$

want f s.t. $\int_0^t W_s\, dW_s = \int_0^t \partial_w f(s, W_s)\, dW_s$

so $\partial_w f(t, w) = w \Rightarrow f(t, w) = \frac{1}{2} w^2 + \underline{l(t)}$

$\hookrightarrow 0$ choose.

then $\partial_t f = 0$, $\partial_{ww} f = 1$

Ito's lemma $\Rightarrow$

$$X_t - X_0 = \int_0^t \Big(\overbrace{0}^{\partial_t f(s, W_s)} + \frac{1}{2} \cdot \overbrace{1}^{\partial_{ww} f(s, W_s)} \Big)\, ds$$

$$+ \int_0^t W_s\, dW_s$$

$$+\int_0^t W_s \, dW_s$$

$\partial_\omega f(s, W_s)$

$$\Rightarrow \int_0^t W_s \, dW_s = X_t - X_0 - \tfrac{1}{2} t$$

$$= f(t, W_t) - f(0, W_0) - \tfrac{1}{2} t$$

$$= \tfrac{1}{2} W_t^2 - \tfrac{1}{2} t$$

approach 2) show that $d\left(\tfrac{1}{2} W_t^2 - \tfrac{1}{2} t\right) = W_t \, dW_t$

$$f(t, \omega) = \tfrac{1}{2}\omega^2 - \tfrac{1}{2} t, \quad X_t = f(t, W_t) = \tfrac{1}{2} W_t^2 - \tfrac{1}{2} t$$

$$dX_t = \left(\partial_t f(t, W_t) + \tfrac{1}{2} \partial_{\omega\omega} f(t, W_t)\right) dt$$

$$+ \partial_\omega f(t, W_t) \, dW_t$$

$$= \left[-\tfrac{1}{2} + \tfrac{1}{2} \cdot 1\right] dt + W_t \, dW_t$$

$$= W_t \, dW_t$$

$$\int_0^t W_s^2 \, dW_s = ?$$

$$\int_0^t \partial_\omega f(s, W_s) \, dW_s \qquad \text{choose } f(t, \omega) = \tfrac{1}{3}\omega^3$$

$$\partial_t f = 0, \quad \partial_{\omega\omega} f = 2\omega \qquad X_t = f(t, W_t)$$

$\partial_t f = 0$, $\partial_{ww} f = 2w$ $X_t = f(t, W_t)$

Ito's lemma $\Rightarrow$

$$X_t - X_0 = \int_0^t (\underbrace{0}_{\partial_t f} + \tfrac{1}{2} \underbrace{2W_s}_{\partial_{ww} f}) \, ds$$

$$+ \int_0^t \underbrace{W_s^2}_{\partial_w f} \, dW_s$$

$$\Rightarrow \boxed{\begin{aligned} \int_0^t W_s^2 \, dW_s &= X_t - X_0 - \int_0^t W_s \, ds \\ &= \tfrac{1}{3} W_t^3 - \int_0^t W_s \, ds \end{aligned}}$$

approach 2) show $d(\text{l.h.s}) = d(\text{r.h.s})$

$$\text{r.h.s.} = \tfrac{1}{3} W_t^3 - \int_0^t W_s ds$$

Ito does not apply (b/c not $f(t, W_t)$)

but $d\left(\int_0^t W_s ds \right) = W_t \, dt$

so choose $f(t, w) = \frac{1}{3} w^3$, $Y_t = f(t, W_t)$

$$dY_t = (\underbrace{0}_{\partial_t f} + \tfrac{1}{2} \underbrace{2W_t}_{\partial_{ww} f}) \, dt + \underbrace{W_t^2}_{\partial_w f} \, dW_t$$

$$\Rightarrow \quad d(\text{rhs}) = \left(W_t\,dt + W_t^2\,dW_t\right) - W_t\,dt$$

$$= W_t^2\,dW_t = d(\text{lhs})$$

$$X_t = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$$

find the SDE that X satisfies.

$$= f(t, W_t), \quad f(t,w) = e^{-\frac{1}{2}\sigma^2 t + \sigma w}$$

$$\partial_t f = -\frac{1}{2}\sigma^2\, e^{-\frac{1}{2}\sigma^2 t + \sigma w}$$

$$\partial_w f = \sigma\, e^{-\frac{1}{2}\sigma^2 t + \sigma w}$$

$$\partial_{ww} f = \sigma^2\, e^{-\frac{1}{2}\sigma^2 t + \sigma w}$$

$$dX_t = \Big[\underbrace{-\frac{1}{2}\sigma^2 X_t}_{\partial_t f(t,W_t)} + \frac{1}{2}\underbrace{\sigma^2 X_t}_{\partial_{ww} f(t,W_t)}\Big]\,dt$$

$$+ \underbrace{\sigma X_t}_{\partial_w f(t,W_t)}\,dW_t$$

$$\Rightarrow \quad \boxed{dX_t = \sigma X_t\,dW_t}$$

Now start with $dX_t = \sigma X_t\, dW_t$ and solve for X_t.

[$dx_t = \sigma x_t\, df_t$ Find x_t

$$\frac{dx_t}{x_t} = \sigma\, df_t$$

$$d \ln x_t = \sigma\, df_t$$

$$\ln x_t - \ln x_0 = \sigma (f_t - f_0)$$

$$\Rightarrow \quad x_t = x_0\, e^{\sigma (f_t - f_0)}$$]

$$dX_t = \sigma X_t\, dW_t$$

$$\Rightarrow \quad \frac{dX_t}{X_t} = \sigma\, dW_t$$

$d(\ln X_t) \neq \dfrac{dX_t}{X_t}$ b/c X_t is driven by a B. motion