Tuesday, December 09, 2008
2:26 PM

2. ** Derive the delta and gamma for a digital put and digital call option using the Black-Scholes model.

digital call $\quad \mathbb{Q} = \mathbb{1}_{S_T > K}$

$\mathbb{Q}$-Wiener process

$$V_t^c = e^{-r(T-t)}\,\mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}_{S_T>K}\right] \quad ; \quad \frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t$$

$$= e^{-r(T-t)}\,\mathbb{Q}_t(S_T > K) \quad , \quad Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$= e^{-r(T-t)}\,\mathbb{Q}\left(Z > \frac{\ln(K/S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$= e^{-r(T-t)}\,\Phi(d_-)$$

$$d_- \triangleq \frac{\ln(S_t/K) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\Delta^c = \partial_S V(S,t)$$

$$= e^{-r(T-t)}\,\Phi'(d_-)\,\partial_S d_-$$

$$\Delta^c = e^{-r(T-t)}\,\frac{\Phi'(d_-)}{S\,\sigma\sqrt{T-t}}$$

and $\quad \Phi'(x) = e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$

$$\Gamma^c = \partial_S \Delta^c$$

$$= \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}\left(\frac{1}{S}\,\Phi''(d_-)\cdot\frac{1}{S\,\sigma\sqrt{T-t}} - \frac{1}{S^2}\,\Phi'(d_-)\right)$$

NB: $\quad \Phi''(x) = -x\,e^{-\frac{1}{2}x^2}/\sqrt{2\pi} = -x\,\Phi'(x)$

$$\Rightarrow \quad \Gamma^c = -\frac{e^{-r(T-t)}}{S^2\sigma\sqrt{T-t}}\left(1+\frac{d_-}{\sigma\sqrt{T-t}}\right)\Phi'(d_-)$$

<u>NB:</u> $\quad \mathcal{Q}^c + \mathcal{Q}^p = 1 \Rightarrow \mathcal{Q}^p = 1 - \mathcal{Q}^c$

$$\Rightarrow \quad V_t^p = e^{-r(T-t)} - V_t^c$$

$$\Rightarrow \quad \Delta_t^p = -\Delta_t^c \quad \text{and} \quad \Gamma_t^p = -\Gamma_t^c$$

(b) ** A forward-start asset-or-nothing option which pays the asset at T if the asset price at maturity is above a percentage α of the asset price at time U (where $t < U < T$). That is, $\varphi = S_T\,\mathbb{I}(S_T > \alpha S_U)$.

For $t \le u$...

$$V_t = e^{-r(T-t)}\,\mathbb{E}_t^{\mathbb{Q}}\left[S_T\,\mathbb{1}(S_T > \alpha S_u)\right]$$

$$= e^{-r(T-t)}\,\mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{E}_u^{\mathbb{Q}}\left[S_T\,\mathbb{1}_{S_T > \alpha S_u}\right]\right]$$

now,

$$\mathbb{E}_u^{\mathbb{Q}}\left[S_T\,\mathbb{1}_{S_T > \alpha S_u}\right] \qquad z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$= S_u e^{(r-\frac{1}{2}\sigma^2)(T-u)}\,\mathbb{E}_u^{\mathbb{Q}}\left[e^{\sigma\sqrt{T-u}\,z}\,\mathbb{1}_{z > \underbrace{\frac{\ln\alpha - (r-\frac{1}{2}\sigma^2)(T-u)}{\sigma\sqrt{T-u}}}_{-z^*}}\right]$$

$$= S_u e^{(r-\frac{1}{2}\sigma^2)(T-u)}\int_{-z^*}^{\infty} e^{\sigma\sqrt{T-u}\,z}\,e^{-\frac{1}{2}z^2}\,\frac{dz}{\sqrt{2\pi}}$$

$$= S_u e^{(r-\frac{1}{2}\sigma^2)(T-u)}\int_{-z^*}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{T-u})^2 + \frac{1}{2}\sigma^2(T-u)}\,\frac{dz}{\sqrt{2\pi}}$$

$$= S_u e^{r(T-u)}\,\Phi(z^* + \sigma\sqrt{T-u})$$

$$\therefore\quad V_t = e^{-r(u-t)}\,\Phi(z^* + \sigma\sqrt{T-u})\,\mathbb{E}_t[S_u]$$

$$= S_t\,\Phi(z^* + \sigma\sqrt{T-u})$$

$$\Rightarrow\quad \Delta_t = \Phi(z^* + \sigma\sqrt{T-u})$$

and $\Gamma_t = 0$

For $t \in [u, T]$

$$V_t = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}[S_T \mathbb{1}_{S_T > \alpha S_u}]$$

S_u is not random at time t!

$$= e^{-r(T-t)} S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)}$$

$$\times \mathbb{E}_t^{\mathbb{Q}}\left[e^{\sigma\sqrt{T-t}\, z} \mathbb{1}_{z > \underbrace{\frac{\ln(\alpha S_u/S_t) - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}}_{-\eta^*}} \right]$$

η^* depends on S_t now!

$$= S_t \, \Phi(\eta^* + \sigma\sqrt{T-t})$$

$$\Delta_t = \Phi(\eta^* + \sigma\sqrt{T-t}) + S_t \, \Phi'(\eta^* + \sigma\sqrt{T-t}) \frac{1}{S_t \sigma\sqrt{T-t}}$$

$$= \Phi(\eta^* + \sigma\sqrt{T-t}) + \frac{\Phi'(\eta^* + \sigma\sqrt{T-t})}{\sigma\sqrt{T-t}}$$

$$\Gamma_t = \frac{\Phi'(\eta^* + \sigma\sqrt{T-t})}{S_t \sigma\sqrt{T-t}} + \frac{\Phi''(\eta^* + \sigma\sqrt{T-t})}{\sigma\sqrt{T-t}} \frac{1}{S_t \sigma\sqrt{T-t}}$$

5. Suppose that interest rates follow the Ho-Lee model:

$$dr_t = \alpha_t \, dt + \sigma \, dW_t$$

where α_t is a deterministic function of time and W_t is a $\mathbb{Q}$-Wiener process. Determine each of the following:

(a) ** Bond price at time t of maturity T.

(b) ** The SDE which the bond price satisfies in terms of W_t.

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s \, ds} \right]$$

now $\int_t^T r_s \, ds = \int_t^T \left(r_t + \int_t^s \alpha_u \, du + \sigma \int_0^s dW_u \right) ds$

$$= r_t (T-t) + \int_t^T \int_t^s \alpha_u \, du + \sigma \int_t^T W_s \, ds$$

clearly $\int_t^T r_s ds \sim \mathcal{N}(m, v)$ and

$$m = \int_t^T \int_0^s \alpha_u \, du + (T-t) r_t$$

$$v = \sigma^2 \, \mathbb{E}\left[\int_t^T W_s \, ds \int_t^T W_u \, du \right]$$

$$= 2\sigma^2 \, \mathbb{E}\left[\int_t^T \int_u^T W_s W_u \, ds \, du \right]$$

$$= 2\sigma^2 \int_t^T \int_u^T s \, ds \, du$$

$$= \sigma^2 \frac{(T-t)^3}{3}$$

$$\therefore \quad P_t(T) = e^{-m + \frac{1}{2} v}$$

$$= \exp\left\{ -(T-t) r_t \underbrace{- \int_t^T \int_t^s \alpha_u \, du \, ds + \frac{\sigma^2}{6} (T-t)^3}_{S_t} \right\}$$

b) write $P_t(T) = e^{-(T-t) r_t + b_t}$ using Ito's lemma...

$$\Rightarrow \quad d P_t(T) = \Big(\overbrace{r_t + \partial_t b_t}^{\partial_t} \overbrace{- (T-t)}^{\partial_r} \cdot \alpha_t + \tfrac{1}{2} \sigma^2 \overbrace{(T-t)^2}^{\partial_{rr}} \Big) P_t(T)\, dt$$

$$\underbrace{-(T-t)}_{\partial_r} P_t(T)\, \sigma\, dW_t$$

note $$\partial_t b_t = -\partial_t \int_t^T \int_t^s \alpha_u \, du\, ds - \frac{\sigma^2}{2}(T-t)^2$$

$$= \int_t^t \alpha_u \, du - \int_t^T \left(\partial_t \int_t^s \alpha_u\, du \right) ds - \frac{\sigma^2}{2}(T-t)^2$$

$$= \int_t^T \alpha_t \, ds - \frac{\sigma^2}{2}(T-t)^2$$

$$= \alpha_t (T-t) - \frac{\sigma^2}{2}(T-t)^2$$

$$\Rightarrow \quad \frac{dP_t(T)}{P_t(T)} = r_t\, dt - (T-t)\, \sigma\, dW_t$$

6. ++ Suppose that two traded stocks have price processes X_t and Y_t. Assume they are jointly GBMs, i.e.

$$\frac{dX_t}{X_t} = \mu_x\,dt + \sigma_x\,dW_t^x\;, \qquad \frac{dY_t}{Y_t} = \mu_y\,dt + \sigma_y\,dW_t^y\;, \tag{1}$$

where X_t and Y_t are correlated standard Brownian motions under the $\mathbb{P}$-measure with correlation ρ. Consider a contingent claim f written on the two assets with payoff $\varphi(X_T, Y_T)$ at time T.

(a) Use a dynamic hedging argument to demonstrate that to avoid arbitrage, the price of f must satisfy the following PDE:

$$\begin{cases} \left(\partial_t + r\,x\partial_x + r\,y\partial_y + \frac{1}{2}\sigma_x^2\,x^2\,\partial_{xx} + \frac{1}{2}\sigma_x^2\,x^2\,\partial_{xx} + \rho\sigma_x\sigma_y\,x\,y\,\partial_{xy}\right) f &= r\,f \\ f(T,x,y) &= \varphi(x,y)\;. \end{cases} \tag{2}$$

set up a self-financing strategy:

a_t units of X_t

b_t " " Y_t

c_t " " M_t

-1 " " f_t

so $V_t = a_t X_t + b_t Y_t + c_t M_t - f_t$ & $V_0 = 0$

since self-financing

$dV_t = a_t\, dX_t + b_t\, dY_t + c_t\, dM_t - df_t$

$= a_t (X_t \mu_x + X_t \sigma_x\, dW_t^x)$

$+ b_t (Y_t \mu_y + Y_t \sigma_y\, dW_t^y)$

$+ c_t\, r M_t\, dM_t$

$- (\partial_t f + \mu_x X_t \partial_x f + \mu_y Y_t \partial_y f$

$$+ \tfrac{1}{2}\sigma_x^2 X_t^2 \partial_{xx} f + \tfrac{1}{2}\sigma_y^2 Y_t^2 \partial_{yy} f$$

$$+ \sigma_x \sigma_y X_t Y_t \partial_{xy} f \Big) dt$$

$$- \sigma_x X_t \partial_x f \, dW_t^x - \sigma_y Y_t \partial_y f \, dW_t^y$$

clearly if $a_t = \partial_x f$ and $b_t = \partial_y f$ then

$$dV_t = \Big[a_t X_t \mu_x + b_t Y_t \mu_y + c_t r M_t$$
$$- \Big(\partial_t f + \mu_x X_t \partial_x f + \mu_y Y_t \partial_y f$$
$$+ \tfrac{1}{2}\sigma_x^2 X_t^2 \partial_{xx} f + \tfrac{1}{2}\sigma_y^2 Y_t^2 \partial_{yy} f$$
$$+ \sigma_x \sigma_y X_t Y_t \partial_{xy} f \Big) \Big] dt$$

To avoid arbitrage $dV_t = 0$ (since no uncertainty in increment).

$$\therefore \quad V_t = 0 \quad \therefore \quad c_t M_t = f_t - a_t X_t - b_t Y_t$$

$$\therefore \quad \partial_x f \, X_t \mu_x + \partial_y f \, Y_t \mu_y + r (f_t - \partial_x f \, X_t - \partial_y f \, Y_t)$$
$$= \partial_t f + \mu_x X_t \partial_x f + \mu_y Y_t \partial_y f$$
$$+ \tfrac{1}{2}\sigma_x^2 X_t^2 \partial_{xx} f + \tfrac{1}{2}\sigma_y^2 Y_t^2 \partial_{yy} f$$
$$+ \sigma_x \sigma_y X_t Y_t \partial_{xy} f$$

cancelling terms and realizing this must hold $\forall X_t, Y_t \Rightarrow$

$$\boxed{\begin{aligned}&\partial_t f + r\, x\, \partial_x f + r\, \partial_y f \\ &+ \tfrac{1}{2}\sigma_x^2\, x^2\, \partial_{xx} f + \tfrac{1}{2}\sigma_y^2\, y^2\, \partial_{yy} f \\ &+ \sigma_x \sigma_y\, xy\, \partial_{xy} f = r f\end{aligned}}$$

and at Maturity T,

$$f(T, x, y) = \varphi(x, y)$$

(b) Suppose that the payoff is homogenous, so that $\varphi(x, y) = y\, g(x/y)$ for some function g. An example of such a payoff is the payoff from an exchange option which would have $\varphi(x, y) = (x - y)_+$. By assuming that $f(t, x, y) = y\, h(t, x/y)$, find the PDE which h satisfies and show that the price f can be written in the form

$$f(X_t, Y_t) = Y_t \mathbb{E}_t^{\mathbb{Q}^*}\left[g(U_T)\right] \tag{3}$$

where, $U_t = X_t/Y_t$ and $\overline{X}_t$ satisfies an SDE of the form

$$\frac{dU_t}{U_t} = \sigma_U\, dW_t^*\ ,$$

for some constant σ_U and W_t^* a $\mathbb{Q}^*$ Brownian motion.

if $f(t, x, y) = y\, h(t, x/y) \stackrel{\Delta}{=} y\, h(t, z)$

then, $\partial_t f = y\, \partial_t h$

$$\partial_x f = y\, \frac{1}{y}\, \partial_z h = \partial_z h$$

$$\partial_{xx} f = \frac{1}{y}\, \partial_{zz} h$$

$$\partial_y f = h + y \cdot \left(-\frac{x}{y^2}\right) \partial_z h$$

$$= h - z\, \partial_z h$$

$$\partial_{yy} f = -\frac{x}{y^2}\,\partial_z h + \frac{x}{y^2}\,\partial_z h - z\left(-\frac{x}{y^2}\right)\partial_{zz} h$$

$$= \frac{1}{y}\, z^2\, \partial_{zz} h$$

$$\partial_{xy} f = -\frac{x}{y^2}\,\partial_{zz} h = -\frac{1}{y}\, z\, \partial_{zz} h$$

so then,

$$y\,\partial_t f + r x\,\partial_z h + r y\,(h - z\,\partial_z h)$$

$$+ \frac{1}{2}\sigma_x^2 x^2 \frac{1}{y}\,\partial_{zz} h$$

$$+ \frac{1}{2}\sigma_y^2 y^2 \frac{1}{y} z^2\,\partial_{zz} h$$

$$+ \sigma_x \sigma_y \rho\, x y \left(-\frac{1}{y}\right) z\,\partial_{zz} h = r y h$$

$$\Rightarrow \partial_t f + \frac{1}{2}\underbrace{(\sigma_x^2 + \sigma_y^2 - 2\sigma_x\sigma_y\rho)}_{\sigma^2} z^2\,\partial_{zz} h = 0$$

and since $f(T, x, y) = y\, g(x/y)$

$$\Rightarrow h(T, z) = g(z)$$

Feynman-Kac $\Rightarrow$

$$h(t, Z_t) = \mathbb{E}^{\mathbb{Q}^*}\left[\, g(Z_T) \mid \mathcal{F}_t \right]$$

where $dZ_t = \sigma Z_t\, dW_t^*$

$$\Rightarrow f(t, X_t, Y_t) = Y_t\, \mathbb{E}^{\mathbb{Q}^*}\left[g(Z_T) \mid \mathcal{F}_t \right]$$