ACT 460 / STA 2502 Stochastic Methods for Actuarial Science - Problem Set #3 due Tuesday, Dec 1 at 2pm. Hand in only questions marked with **. Grad students also hand in questions marked with ++

1. Consider each of the following options:

   (a) digital call struck at 100

   (b) digital put struck at 100

   (c) put struck at 100

   (d) call struck at 100

   (e) strangle struck at 100

   (f) straddle with $K_1 = 95$, $K_2 = 115$

   (g) bull spread with $K_1 = 95$, $K_2 = 115$

   Use the `portfolio.xls` file to explore the sensitive of prices, Deltas and Gammas to T, σ, and r.

2. Derive the price, delta and gamma for an asset-or-nothing call option (which pays S_T if $S_T > K$ at maturity T, and pays 0 otherwise) using the Black-Scholes model. Plot the price, delta and gamma as a function of the spot price with the following parameters: $S_0 = 1$, $K = 1$, $\sigma = 20\%$, $r = 2\%$ for maturities of $T = 1/51, 1/12, 1/2$ and 1 year.

3. Using the Black-Scholes model, determine the price, the delta and the gamma for all times $t \in [0, T)$ of the following European options with payoffs at time $T > 0$:

   (a) A forward-start digital call option, which pays 1 at T if the asset price at maturity is above a percentage α of the asset price at time U (where $t < U < T$). That is, $\varphi = \mathbb{I}(S_T > \alpha S_U)$.

   (b) ** A forward-start asset-or-nothing option which pays the asset at T if the asset price at maturity is above a percentage α of the asset price at time U (where $t < U < T$). That is, $\varphi = S_T\,\mathbb{I}(S_T > \alpha S_U)$.

   (c) A call option (maturing at V) on a forward-start asset-or-nothing option. The embedded forward-start asset-or-nothing option pays the asset at $T > V$ if the asset price at T is above a percentage α of the asset price at time U (where $V < U < T$). The strike of the call option is K.

4. Suppose that the price of a stock is modeled as follows:

$$\frac{dS_t}{S_t} = \mu_t\, dt + \sigma_t\, dW_t$$

where μ_t and σ_t are functions only of time and where W_t is a $\mathbb{P}$-Wiener process. Furthermore, assume that the risk-free interest rate r_t is function only of time. Determine the price, the delta and the gamma for each of the following options:

(a) call option maturing at T strike of K.

(b) forward starting put option with strike set to αS_U at time U and maturing at T.

5. Suppose that interest rates follow the Ho-Lee model:

$$dr_t = \theta_t \, dt + \sigma \, dW_t$$

where α_t is a deterministic function of time and W_t is a $\mathbb{Q}$-Wiener process. Determine each of the following:

(a) ** Bond price at time t of maturity T.

(b) The SDE which the bond price satisfies in terms of W_t.

(c) The choice of θ_t which makes the model prices equal the market prices $P_t^*(T)$.

6. ++ Suppose that two traded stocks have price processes X_t and Y_t. Assume they are jointly GBMs, i.e.

$$\frac{dX_t}{X_t} = \mu_x \, dt + \sigma_x \, dW_t^x \, , \qquad \frac{dY_t}{Y_t} = \mu_y \, dt + \sigma_y \, dW_t^y \, , \tag{1}$$

where X_t and Y_t are correlated standard Brownian motions under the $\mathbb{P}$-measure with correlation ρ. Consider a contingent claim f written on the two assets with payoff $\varphi(X_T, Y_T)$ at time T.

(a) Use a dynamic hedging argument to demonstrate that to avoid arbitrage, the price of f must satisfy the following PDE:

$$\begin{cases} \left(\partial_t + r\, x\partial_x + r\, y\partial_y + \frac{1}{2}\sigma_x^2\, x^2\, \partial_{xx} + \frac{1}{2}\sigma_x^2\, x^2\, \partial_{xx} + \rho\sigma_x\sigma_y\, x\, y\, \partial_{xy}\right) f &= r\, f \\ f(T, x, y) &= \varphi(x, y) \; . \end{cases} \tag{2}$$

(b) Suppose that the payoff is homogenous, so that $\varphi(x, y) = y\, g(x/y)$ for some function g. An example of such a payoff is the payoff from an exchange option which would have $\varphi(x, y) = (x - y)_+$. By assuming that $f(t, x, y) = y\, h(t, x/y)$, find the PDE which h satisfies and show that the price f can be written in the form

$$f(X_t, Y_t) = Y_t \mathbb{E}_t^{\mathbb{Q}^*}\left[g(U_T)\right] \tag{3}$$

where, $U_t = X_t/Y_t$ and $\overline{X}_t$ satisfies an SDE of the form

$$\frac{dU_t}{U_t} = \sigma_U \, dW_t^* \, ,$$

for some constant σ_U and W_t^* a $\mathbb{Q}^*$ Brownian motion.