UNIVERSITY OF TORONTO

Faculty of Arts and Science

Term Test, October 16th, 2012

ACT 460 / STA 2502

DURATION - 120 minutes

EXAMINER: Prof. S. Jaimungal

COURSE CODE (**circle one**): ACT 460 STA 2502

LAST NAME: Solutions

FIRST NAME: ____________________

STUDENT #: ____________________

Each question is worth 10 points

– NOT ALL QUESTIONS ARE OF THE SAME DIFFICULTY .

Please write clearly!

NO AIDS ALLOWED – NO CALCULATORS

Recall that $\mathbb{E}[e^{uZ}] = e^{\frac{1}{2}u^2}$ where $Z \sim \mathcal{N}(0,1)$ is a standard normal random variable.

1 *[10]*	2 *[10]*	3 *[10]*	4 *[10]*	5 *[10]*	6 *[10]*	Total *[60]*

1. Write down concise (about 50 words) but precise responses to each of the following:

   (a) *[5]* What is the fundamental theorem of asset pricing?

$\exists$ $\mathbb{Q}$ s.t. $C_0 = \cancel{e^{-rT}}\, \mathbb{E}^{\mathbb{Q}}[e^{-rT} C_1]$

$\forall$ traded C

$\Longleftrightarrow$ $\nexists$ arb.

   (b) *[5]* What is unacceptable about the Ho-Lee model of interest rates?

it is normally distributed and

it's variance grows with T

$\Rightarrow$ high prob of $r_T < 0$.

2. *[10]* Please indicate true or false. **no explanations required**

-1 for incorrect answer, $+2$ for correct answer, 0 for blank answer .

(a) [T] [F]

The price of a call option always decreases with increasing volatility.

(b) [T] [F]

If an interest rate model matches bond prices at all maturities, then the risk-neutral branching probabilities must equal $\frac{1}{2}$.

(c) [T] [F]

Suppose a risk-neutral measure exists, then the price of traded assets are unique.

(d) [T] [F]

The limiting distribution of a discrete time asset price model must always be log-normal.

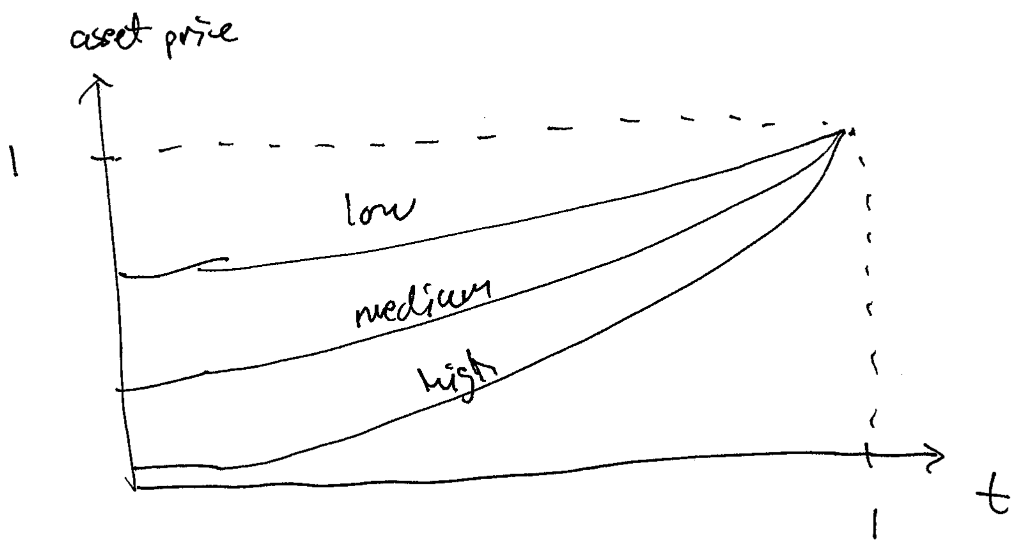
(e) [T] [F]

A forward start call option price is linear in the current spot price.

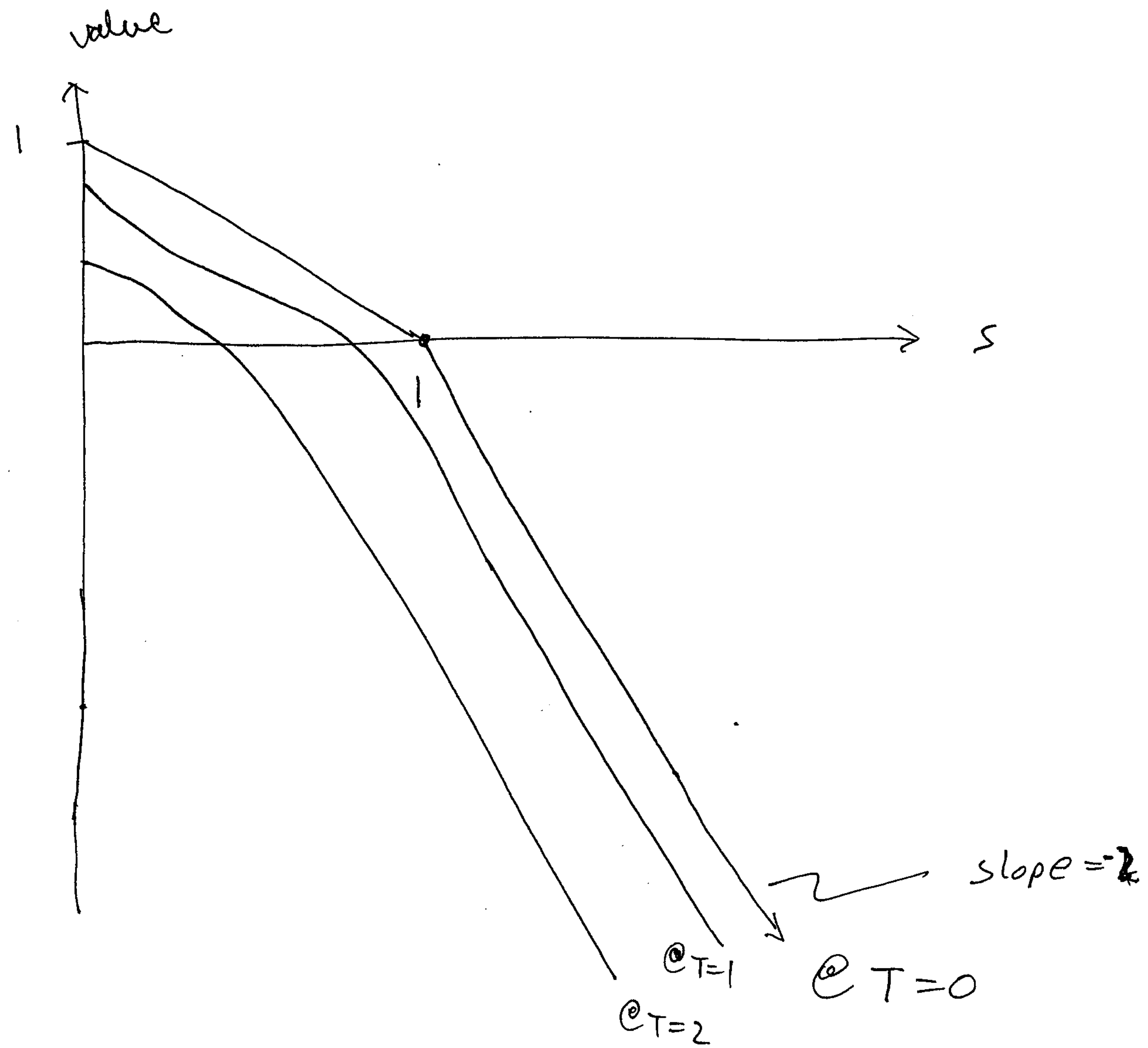
3. Sketch each of the following:

(a) *[5]* The optimal exercise curves for an American put option with maturity $T = 1$ and strike $K = 1$ for three levels of volatility (i) low (ii) medium and (iii) high.

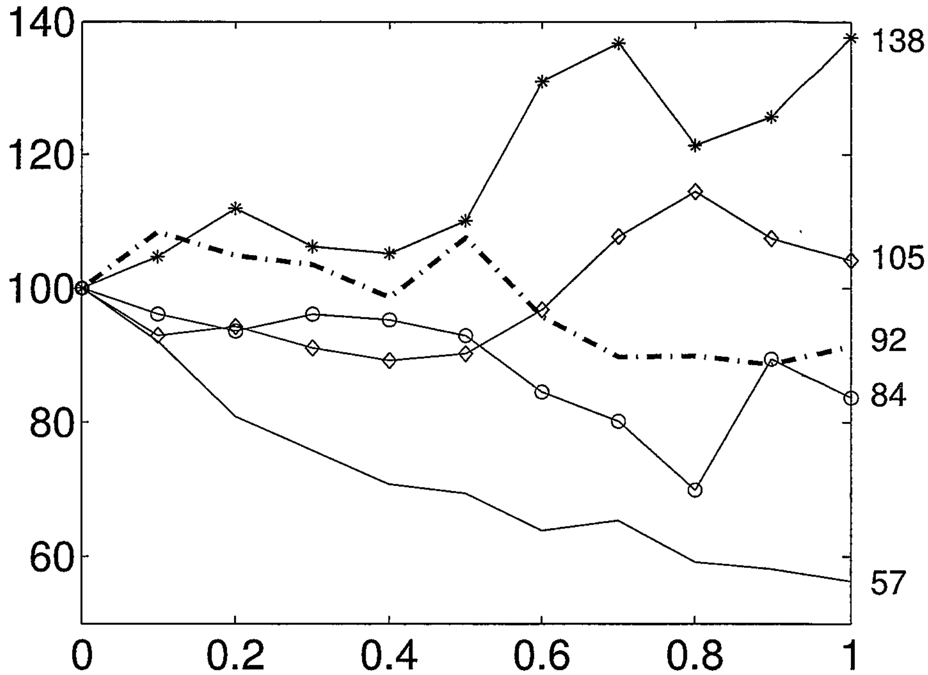
[sketch the three curves on the same graph, clearly label them and any interesting points.]



(b) *[5]* The price of a portfolio consisting of 1 long put struck at \$1, and 2 short calls struck at \$1 for three maturities: (i) at maturity $T = 0$ (ii) maturity $T = 1$ and (iii) maturity $T = 2$. *[sketch the three curves on the same graph, clearly label them and any interesting points.]*



4. (a) You are given that the following 5 paths are the only possible paths of an asset price and that their risk-neutral probabilities are all $\frac{1}{5}$ and $r = 0$.



Write down values (don't carry out the arithmetic) for each of the following 1-year options

i. *[1]* a European call struck at 100.

$$\frac{5+38}{5}$$

ii. *[1]* a European Binary put struck at 110.

$$\frac{4}{5}$$

iii. *[1]* a down-and-in knock-in call with lower barrier of 80 and a strike of 70.

$$\frac{14}{5}$$

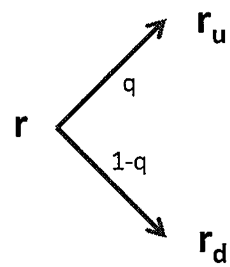
iv. *[1]* an up-and-out knock-out put with upper barrier of 110 and a strike of 100.

$$\frac{8 + 16 + 43}{5}$$

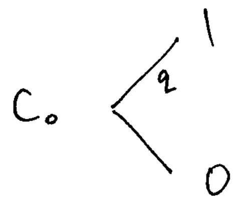
v. *[1]* an up-and-in knock-in binary put with upper barrier of 105 and strike 110.

$$\frac{2}{5}$$

(b) *[5]* Consider the interest rate tree shown diagram below ($r_u > r_d$) – each time step is 1-year.

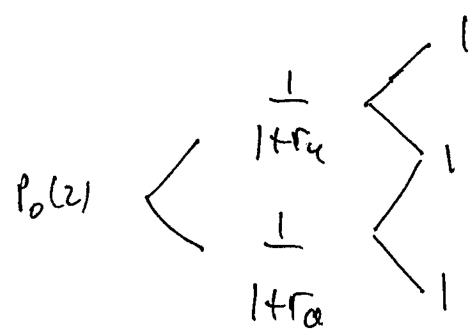


The rates correspond to effective discounting – e.g. discounting over the first period is $1/(1+r)$. You are told that a contingent claim maturing at $t = 1$ paying 1 if the interest rates rise and 0 if they drop has value $C_0 = \frac{1}{4}\frac{1}{1+r}$. Derive an expression for the value of a 2-year zero coupon bond in terms of r, r_u and r_d only.



$$\Rightarrow \quad C_0 = \frac{q}{1+r} = \frac{1}{4}\,\frac{1}{1+r}$$

$$\Rightarrow \quad q = \frac{1}{4}$$



$$P_0(2) = \frac{1}{1+r}\left[\frac{q}{1+r_u} + \frac{1-q}{1+r_d}\right] = \frac{1}{1+r}\left[\frac{1}{4(1+r_u)} + \frac{3}{4(1+r_d)}\right]$$

5. Assume an equity price S_t is modeled as in the Black-Scholes model (i.e. the limiting case of the CRR model as $\Delta t \downarrow 0$ and interest rates are constant at r). For each of the following, write your answers terms of $\Phi(x) \triangleq \mathbb{Q}(Z < x)$ where Z is a standard normal random variable under the risk-neutral measure $\mathbb{Q}$.

   (a) *[5]* **Derive** an expression for the ($t = 0$) price of an option with T-maturity payoff

$$\varphi = S_T \, \mathbb{I}_{S_T<K} \, .$$

   Here K is a constant and, as usual, $\mathbb{I}_\omega$ is the indicator function of the event ω, i.e. equals 1 if ω occurs and 0 otherwise.

$$V_0 = e^{-rT} \mathbb{E}^{\mathbb{Q}}[\varphi]$$

$$= e^{-rT} \underbrace{\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{S_T<K}]}_{\mathcal{E}}$$

and $S_T \stackrel{d}{=} S_0 \, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,z}$, $\quad z \underset{\mathbb{Q}}{\sim} N(0,1)$

$$\Rightarrow \mathcal{E} = \mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{S_T<K}] = \mathbb{E}\Big[S_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,z} \times \mathbb{1}_{z<z^*}\Big]$$

where $z^* = \dfrac{\ln(K/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$

$$\Rightarrow \quad \mathcal{E} = S_0\, e^{(r-\frac{1}{2}\sigma^2)T} \underbrace{\mathbb{E}^{\mathbb{Q}}[e^{\sigma\sqrt{T}\,z}\, \mathbb{1}_{z<z^*}]}_{F}$$

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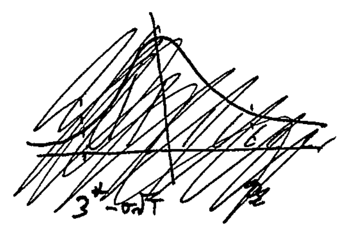
$$f = \int_{-\infty}^{\infty} e^{\sigma\sqrt{T} z} \, \mathbb{1}_{z < z^*} \, \frac{e^{-\frac{1}{2} z^2}}{\sqrt{2\pi}} \, dz$$

$$= \int_{-\infty}^{z^*} e^{\sigma\sqrt{T} z - \frac{1}{2} z^2} \frac{dz}{\sqrt{2\pi}}$$

$$= \int_{-\infty}^{z^*} e^{-\frac{1}{2}(z - \sigma\sqrt{T})^2 + \frac{1}{2}\sigma^2 T} \frac{dz}{\sqrt{2\pi}}$$

$$= e^{\frac{1}{2}\sigma^2 T} \int_{-\infty}^{z^* - \sigma\sqrt{T}} \frac{e^{-\frac{1}{2} y^2}}{\sqrt{2\pi}} \, dy$$



$$= e^{\frac{1}{2}\sigma^2 T} \, \Phi\left(+ z^* - \sigma\sqrt{T}\right)$$

$$\Rightarrow \quad \mathcal{E} = S_0 \, e^{rT} \, \Phi\left(z^* - \sigma\sqrt{T}\right)$$

good enough.

$$\Rightarrow \quad V_0 = S_0 \, \Phi\Bigg(\overbrace{- \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}}^{z^* - \sigma\sqrt{T}} \Bigg)$$

$$= S_0 \, \Phi\left(- \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

(b) *[5]* **Derive** an expression for the $(t=0)$ price of a forward starting option with T-payoff

$$\varphi = \max(S_T, \alpha S_U) - S_V .$$

Here, $0 < V < U < T$, $\alpha > 0$ is a constant.

$$C_0 = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}[\varphi]$$

$$= e^{-rT}\left(\mathbb{E}^{\mathbb{Q}}[\max(S_T, \alpha S_u)] - \mathbb{E}^{\mathbb{Q}}[S_V]\right)$$

$$= e^{-rT}\, \underbrace{\mathbb{E}^{\mathbb{Q}}[\max(S_T, \alpha S_u)]}_{h} - e^{-rT} \cdot e^{rV} S_0$$

$$h = \mathbb{E}^{\mathbb{Q}}\Big[\underbrace{\mathbb{E}^{\mathbb{Q}}[\max(S_T, \alpha S_u) \mid S_u]}_{g}\Big]$$

$$g = \mathbb{E}^{\mathbb{Q}}\left[S_T\, \mathbb{1}_{S_T \geq \alpha S_u} + \alpha S_u\, \mathbb{1}_{S_T < \alpha S_u} \mid S_u\right]$$

now $\quad S_T \overset{d}{=} S_u\, e^{\overbrace{(T-u)}^{\tau}(r-\frac{1}{2}\sigma^2) + \sigma\sqrt{T-u}\, Z}, \quad Z \underset{\mathbb{Q}}{\sim} N(0,1)$

$$\Rightarrow g_1 = \mathbb{E}^{\mathbb{Q}}[S_T\, \mathbb{1}_{S_T \geq \alpha S_u} \mid S_u] = S_u\, e^{(r-\frac{1}{2}\sigma^2)\tau} \cdot \mathbb{E}^{\mathbb{Q}}\left[e^{\sigma\sqrt{\tau} Z} \cdot \mathbb{1}_{Z \geq z^*}\right]$$

where z^* s.t. $\quad S_u\, e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\, z^*} = \alpha S_u$

$$\Rightarrow z^* = \frac{\ln \alpha - (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$\Rightarrow$ ~~$\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}$~~ $\quad\quad \hookrightarrow$ independent of S_u!

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hence,

$$g_1 = S_u e^{(r-\frac{1}{2}\sigma^2)\tau} \int_{z^*}^{+\infty} e^{\sigma\sqrt{\tau} z} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz$$

$$= S_u e^{(r-\frac{1}{2}\sigma^2)\tau} \int_{z^*}^{+\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{\tau})^2 + \frac{1}{2}\sigma^2\tau} dz$$

$$= S_u e^{r\tau} \quad \Phi(-z^* + \sigma\sqrt{\tau})$$

and $g_2 = \alpha S_u \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{S_T < \alpha S_u} | S_u]$

$$= \alpha S_u \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{Z < z^*}]$$

$$= \alpha S_u \Phi(z^*)$$

$$\Rightarrow \quad h = \mathbb{E}^{\mathbb{Q}}[S_u e^{r\tau}\Phi(-z^*+\sigma\sqrt{\tau}) + \alpha S_u \Phi(z^*)]$$

$$= S_0 e^{ru}[e^{r\tau}\Phi(-z^* + \sigma\sqrt{\tau}) + \alpha \Phi(z^*)]$$

$$\Rightarrow C_0 = S_0\left(\Phi(-z^* + \sigma\sqrt{\tau}) + e^{-r(T-u)}\alpha\Phi(z^*)\right) - e^{-r(T-v)} S_0$$

6. Consider the (discrete) model of short rate of interest r_{t_n} (at time step n, $t_n = n\Delta t$) given recursively by

$$r_{t_n} = r_{t_{n-1}} + \sigma\sqrt{\Delta t}\, x_n$$

where x_n are ~~iid~~ independent (± 1) Bernoulli r.v. with $\mathbb{Q}(x_k = 1) = q_k$

(a) Determine q_k such that r_{t_n} has a drift of $\theta_{t_{n-1}}$, i.e. such that $\mathbb{E}^{\mathbb{Q}}[r_{t_n} - r_{t_{n-1}}] = \theta_{t_{n-1}}\Delta t$ and find the limiting distribution of r_T where $T = N\Delta t$ as $N \to +\infty$.

$$\mathbb{E}^{\mathbb{Q}}[r_{t_n} - r_{t_{n-1}}] = \sigma\sqrt{\Delta t}\,(+1\, q_n + (-1)(1-q_n))$$

$$= \sigma\sqrt{\Delta t}\,(2q_n - 1) = \theta_{t_{n-1}}\Delta t$$

$$\Rightarrow \boxed{q_n = \frac{1}{2}\left[1 + \frac{\theta_{t_{n-1}}\sqrt{\Delta t}}{\sigma}\right]}$$

clearly by CLT we'll have $r_T \xrightarrow[N\to+\infty]{d} \mathcal{N}(m, v)$, so need m and v.

$$\mathbb{E}^{\mathbb{Q}}[r_{t_N}] = \sigma\sqrt{\Delta t}\sum_{n=1}^{N}\mathbb{E}^{\mathbb{Q}}[x_n]$$

$$= \sum_{n=1}^{N}\theta_{t_{n-1}}\Delta t \longrightarrow \boxed{\int_0^T \theta_s\, ds = m}$$

$$\mathbb{V}^{\mathbb{Q}}[r_{t_N}] = \sigma^2\Delta t\sum_{n=1}^{N}\mathbb{V}^{\mathbb{Q}}[x_n]$$ since $x_1, x_2, \ldots$ are independent

$$\mathbb{V}^{\mathbb{Q}}[x_n] = \mathbb{E}[x_n^2] - (\mathbb{E}[x_n])^2$$

$$= 1 - (\theta_{t_{n-1}}\Delta t)^2$$

$$\Rightarrow \mathbb{V}^{\mathbb{Q}}[r_{t_N}] = \sigma^2\Delta t\left(N - \left(\sum_{n=1}^{N}\theta_{t_{n-1}}\Delta t\right)\Delta t\right) \longrightarrow \boxed{\sigma^2 T = v}$$

(b) *[6]* Determine the limiting (as $N \to +\infty$) joint distribution of r_T and $I_T = \int_0^T r_s\, ds$ where $T = N\Delta t$.

$$I_T = \sum_{n=1}^{N} r_{t_{n-1}} \Delta t = \sum_{n=1}^{N} \sum_{m=1}^{n-1} x_m \sigma \Delta t^{3/2}$$

$$\text{so } \mathbb{E}[I_T] = \cancel{\sum_{n=1}^{N} \theta_{t_{n-1}} \Delta} - \sum_{n=1}^{N} \sum_{m=1}^{n-1} (2q-1)\, \sigma \Delta t^{3/2}$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{n-1} \theta_{t_{n-1}} \Delta t^2 \longrightarrow \int_0^T \int_0^{s} \theta_u \, du\, ds$$

for $\mathbb{V}[I_T]$ note that:

$$\sum_{n=1}^{N} \sum_{m=1}^{n-1} x_m = (0 + x_1 + x_1 + x_2 + \dots + x_1 + x_2 + x_3 + \dots + x_{N-1}) = (N-1)x_1 + (N-2)x_2 + \dots + x_{N-1}$$

$$\Rightarrow \mathbb{V}[I_T] = \sigma^2 \Delta t^3 \sum_{n=1}^{N-1} (N-n)^2\, \mathbb{V}[x_n] = \sum_{n=1}^{N-1} (N-n)\, \sigma^2 \Delta t^3 - \underbrace{\sum_{n=1}^{N-1} (N-n)\theta_{t_{n-1}} \Delta t \cdot \sigma^2 \Delta t^4}_{\to 0 \text{ as } N \to +\infty}$$

x_n are independent

now (we already have $\mathbb{V}[x_n] = 1 - (\theta_{t_{n-1}} \Delta t)^2$)

$$\Rightarrow \mathbb{V}[I_T] \to \lim_{N \to +\infty} \sigma^2 \Delta t^3 \sum_{n=1}^{N-1} (N-n) = \lim_{N\to\infty} \frac{\sigma^2 \Delta t^3}{6} \cdot (2N-1)(N-1)N$$

$$= \frac{\sigma^2 T^3}{3}$$

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next need $\mathbb{C}[\,r_T, I_T)$

$$= \mathbb{C}\Big[\sum_{n=1}^{N} x_n \sigma \sqrt{\Delta t}\,,\ \sum_{n=1}^{N-1} (N-n)\, x_n\, \sigma \Delta t^{3/2}\Big]$$

$$= \sigma^2 \Delta t^2 \sum_{n=1}^{N-1} \mathbb{C}[x_n, x_n]\,(N-n)$$

$$= \sigma^2 \Delta t^2 \sum_{n=1}^{N-1} (1 - \theta_{t_{n-1}} \Delta t)(N-n)$$

$$= \sigma^2 \Delta t^2 \cdot \frac{N(N-1)}{2} - \sigma^2 \Delta t^2 \sum_{n=1}^{N-1} (N-n)\, \theta_{t_{n-1}} \Delta t$$

$$\to \frac{\sigma^2 T^2}{2}$$

so by CLT

$$\begin{pmatrix} r_T \\ I_T \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \int_0^T \theta_u\, du \\ \int_0^T \int_0^s \theta_u\, du\, ds \end{pmatrix} ; \begin{pmatrix} \sigma^2 T & \frac{\sigma^2 T^2}{2} \\ \frac{\sigma^2 T^2}{2} & \frac{\sigma^2 T^3}{3} \end{pmatrix}\right)$$