First Passage Times: Integral Equations, Randomization and Analytical Approximations

by

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A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Department of Statistics
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Abstract

The first passage time (FPT) problem for Brownian motion has been extensively studied in the literature. In particular, many incarnations of integral equations which link the density of the hitting time to the equation for the boundary itself have appeared. Most interestingly, Peskir (2002b) demonstrates that a master integral equation can be used to generate a countable number of new integrals via its differentiation or integration. In this thesis, we generalize Peskir’s results and provide a more powerful unifying framework for generating integral equations through a new class of martingales. We obtain a continuum of new Volterra type equations and prove uniqueness for a subclass. The uniqueness result is then employed to demonstrate how certain functional transforms of the boundary affect the density function.

Furthermore, we generalize a class of Fredholm integral equations and show its fundamental connection to the new class of Volterra equations. The Fredholm equations are then shown to provide a unified approach for computing the FPT distribution for linear, square root and quadratic boundaries. In addition, through the Fredholm equations, we analyze a polynomial expansion of the FPT density and employ a regularization method to solve for the coefficients.

Moreover, the Volterra and Fredholm equations help us to examine a modification of the classical FPT under which we randomize, independently, the starting point of the Brownian
motion. This randomized problem seeks the distribution of the starting point and takes
the boundary and the (unconditional) FPT distribution as inputs. We show the existence
and uniqueness of this random variable and solve the problem analytically for the linear
boundary. The randomization technique is then drawn on to provide a structural framework
for modeling mortality. We motivate the model and its natural inducement of 'risk-neutral’
measures to price mortality linked financial products.

Finally, we address the inverse FPT problem and show that in the case of the scale family
of distributions, it is reducible to finding a single, base boundary. This result was applied
to the exponential and uniform distributions to obtain analytical approximations of their
corresponding base boundaries and, through the scaling property, for a general boundary.
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## Contents

1 Introduction

1.1 FPT ................................. 6
   1.1.1 Upper and Lower Boundaries .... 7
   1.1.2 Integral Equations ............... 8
   1.1.3 Small Time Behavior of the FPT Density . 13
   1.1.4 PDE Approach ...................... 15

1.2 Inverse FPT .......................... 16

1.3 Main Results and Outline .......... 19

2 Integral Equations ................... 25

2.1 Volterra Integral Equations ......... 25
   2.1.1 Passage to the limit ............. 32
   2.1.2 Special Cases ..................... 37
   2.1.3 Uniqueness of a solution ......... 40
   2.1.4 Functional Transforms .......... 47

2.2 Fredholm Equations .................. 50

3 Randomized FPT ....................... 60

3.1 Uniqueness and Existence .......... 62

3.2 Linear Boundary ..................... 73
3.2.1 Back to the Classical FPT

4 Mortality Modeling with Randomized Diffusion
   4.1 The Model and the Fit
   4.2 Risk Neutral Pricing of Mortality Linked Securities

5 Approximate Analytical Solutions to the FPT and IFPT Problems
   5.1 An Application of the Method of Images for a Class of Boundaries
   5.2 Polynomial Expansion of FPT density
   5.3 Space and Time Change

6 Conclusion

A Supplementary Results
   A.0.1 Parabolic Cylinder Function
   A.0.2 Airy function

Bibliography
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>A sample path of the stopped Brownian motion $W(t \wedge \tau)$</td>
<td>3</td>
</tr>
<tr>
<td>2.1</td>
<td>The contours of integration for quadratic boundaries in Example 3.</td>
<td>57</td>
</tr>
<tr>
<td>3.1</td>
<td>A sample path of the randomized Brownian motion $W(t) + X$</td>
<td>61</td>
</tr>
<tr>
<td>4.1</td>
<td>The kernel estimator of the distribution of the hitting time using Dirac measures for the randomized starting health unit.</td>
<td>88</td>
</tr>
<tr>
<td>4.2</td>
<td>The model fit to the Sweedish cohort data. Panel (a) shows the life table data fitted with a mixture of Gamma distributions using the kernel estimator (4.7) with $v = 3^2$. Panel (b) compares the distribution of the hitting time with that of the initial level using a volatility $\beta = 0.95, \beta_{\text{max}} = 19.3%$.</td>
<td>90</td>
</tr>
<tr>
<td>5.1</td>
<td>a) Numerical and Laguerre density and cdf; b) Difference of densities and cdf’s</td>
<td>108</td>
</tr>
<tr>
<td>5.2</td>
<td>a) Numerical and Laguerre density and cdf; b) Difference of densities and cdf’s</td>
<td>108</td>
</tr>
<tr>
<td>5.3</td>
<td>a) Numerical and Laguerre density and cdf; b) Difference of densities and cdf’s</td>
<td>109</td>
</tr>
<tr>
<td>5.4</td>
<td>a) Numerical and Laguerre density and cdf; b) Difference of densities and cdf’s</td>
<td>109</td>
</tr>
<tr>
<td>5.5</td>
<td>a) $U[0,1]$ boundary; b) Difference of $U[0,1]$ boundary and $\hat{b}_1$</td>
<td>115</td>
</tr>
<tr>
<td>5.6</td>
<td>a) $Exp(1)$ boundary on $[0,3]$; b) Difference between $Exp(1)$ boundary and $\hat{b}_1$</td>
<td>116</td>
</tr>
<tr>
<td>5.7</td>
<td>a) Difference between $U(0,\lambda)$ and $\hat{b}<em>\lambda$; b) Difference between $Exp(\lambda)$ and $\hat{b}</em>\lambda$.</td>
<td>117</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

The motivation for studying first passage time problems is two-fold. On the one hand, they are of great theoretical interest since they are connected to many fields of mathematics such as probability theory, functional and numerical analysis, statistics and optimal control. On the other hand, their potential applicability has drawn tremendous amount of attention in many scientific disciplines. In a variety of problems related to applications in biology, chemistry, astrophysics (Zhang and Hui (2006)), engineering and mathematical psychology (Holden (1976), Ricciardi (1977)) one faces the evaluation of objects arising from first passage time probabilities. For instance the extinction of a population can, at times, be described as the first passage through some threshold value for the process representing the number of individuals; the firing of a neuron may be depicted as the first crossing of some threshold value by the process modeling the membrane potential difference. In quantitative finance such questions arise in many practical issues such as the pricing of barrier options and credit risk. Barrier options have become increasingly popular hedging and speculation tools in recent years. These options embed digital options. If the relative position of the underlying and boundary matters at the date of maturity, these binary derivatives are of the European type and their valuation is simpler. If the relative position matters during the entire time
to maturity, pricing these digital derivatives is more involved as they are path dependent. In the latter case, they are dubbed one-touch digital options and their valuation boils down to computing first passage time distributions. In the pricing of credit derivatives the main ingredient is modeling the risk associated with time until ‘default’ or the inability of a company to meet its financial obligations. This time till default can be viewed as a first passage time to a time-dependent boundary of a process representing the credit worthiness of the company. In statistical science, the grandfather of all such problems is to determine the distribution of the one-sample Kolmogorov-Smirnov statistic which is the first passage time of a process $X_t$ to a constant boundary. Here $X_t$ is the difference between the empirical and true distribution function of a random sample. A similar statistic was proposed by Anderson and Darling (1952). They observed that the limiting distribution of this statistic is the same as the distribution of the first passage time of a Brownian bridge to a constant boundary. The principal contemporary motivation for studying such problems in the field of statistics comes from sequential analysis. For example, repeated significance test is a sequential test designed to stop sampling as soon as it is apparent that $H_1$ is true while behaving like a fixed sample test if $H_0$ appears to be true. The time to stop sampling is essentially a first passage time (see Siegmund (1986)).

There are three ingredients in the setting of the classical FPT problems. Namely, the boundary $b(t)$, which is normally assumed to be a smooth function of time; the diffusion process $X_t$ and the distribution, $F(t)$, of the first time the process crosses the boundary (if ever). The forward, first passage time, problem (FPT) seeks the distribution $F(t)$ assuming $X_t$ and $b(t)$ as given. In the inverse, first passage time, (IFPT) problem we are interested in $b(t)$ given $X_t$ and $F(t)$. In both problems the process dynamics is assumed to be known.

More formally, let $X_t$ be the solution to the following stochastic differential equation

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW, \quad X_0 = x \geq b(0)$$
where $W_t$ is a standard Brownian motion on a probability space equipped with the filtration $F_t$ and $\mu : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and $\sigma : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ are smooth bounded functions. We define the first passage time of the diffusion process $X_t$ to the curved boundary $b(t)$ to be:

$$\tau = \inf\{t > 0; X_t \leq b(t)\}$$

(1.1)

and $F(t) := \mathbb{P}(\tau \leq t)$. Note that $\tau$ may have a positive probability mass at $t = \infty$; that is the case when $\tau$ never crosses the boundary. Also, notice that, since $X_t$ has continuous paths, at $\tau$, the process and the boundary have the same value. However, in the definition of $\tau$ we use the symbol '$\leq$' instead of '$=$' to define $\tau$ as the first passage time 'from above' to the boundary (see Fig. 1.1 below).

![Figure 1.1: A sample path of the stopped Brownian motion $W(t \wedge \tau)$](image)

The two classical first-passage time problems are formulated as follows.

The **FPT problem**: Given a boundary function $b(t)$, find the probability $F(t)$ that $X$ crosses $b$ before or at time $t$.

The **IFPT problem**: Given a cumulative distribution function, $F(t)$, find a boundary function $b(t)$ such that $\mathbb{P}(\tau \leq t) = F(t)$

While the focus of this thesis is on the case $(\mu, \sigma) = (0, 1)$, i.e. $X_t = W_t$ is a standard
Brownian motion, we will also investigate first passage times of time/space changed Brownian
motions of the form \( g(W_{B(t)}, t) \) where the time change \( B(t) \) is a positive monotone increasing
function and the space change \( g(w, \cdot) \) is a monotone function. For this class of diffusions we
can reduce the first passage time problems to those corresponding to the standard Brownian
motion by inverting the time/space change. In the case \( X_t = W_t \), definition (1.1) is equivalent
to \( \hat{\tau} = \inf\{t > 0; \hat{W}_t \geq -b(t)\} \) where \( \hat{W}_t = -W_t \) is also a Brownian motion, since \( \hat{\tau} = \tau \)
a.s.. Thus, \( \hat{\tau} \) is the first passage time 'from below' of a standard Brownian motion to the
boundary \( \hat{b}(t) := -b(t) \). This symmetric property of the Brownian motion implies that in
the FPT problem the boundaries \( \hat{b}(t) \) and \( b(t) \) result in the same distribution, \( F(t) \), for \( \hat{\tau} \)
and \( \tau \) respectively.

The main tools for attacking the first passage time problems are partial differential equa-
tions (PDE), space and time change, measure change and the martingale approach via the
optional sampling theorem. For the FPT problem the formulation in the PDE setting is
done using the Kolmogorov forward equation. Define \( w(x, t) = \mathbb{P}(X_t \leq x, \tau > t) \) and let
\( u(x, t) = dw/dx \). From standard results in probability theory the function \( w(x, t) \) satisfies
the Kolmogorov forward equation

\[
\frac{d}{dt}u(x,t) = \frac{1}{2} \sigma^2 u_{xx} - \mu u_x \text{ for } x > b(t), \; t > 0
\]  

(1.2)

with boundary and initial conditions

\[
u(x, t) = 0 \text{ for } x \leq b(t), \; t > 0 \]  

(1.3)

\[
u(x, 0) = \delta(x) \text{ for } x > 0, \; t = 0 \]  

(1.4)

where \( \delta \) is a Dirac measure at 0. Given sufficiently regular \( b \), this system has a unique
solution and \( \mathbb{P}(\tau > t) = \int_{b(t)}^{\infty} u(x, t)dx, \forall t \geq 0. \) Alternatively, \( f(t) = (1/2)(\sigma^2 u)_x|x=b(t) \). The
PDE approach is used by Lerche (1986) to construct analytic results for a certain class of
boundaries. In Chapter 5, we will apply this methodology to a class of boundaries which to our knowledge has not been investigated in the context of FPT.

The space/time change approach is mostly useful for reduction of processes of the form $X_t = g(W_{B(t)}, t)$ to a Brownian motion after the inversion of $g$ and $B$. The simplest examples of such processes are the Ornstein-Uhlenbeck process, Brownian bridge and geometric Brownian motion. Continuous Gauss-Markov processes are also a subclass of space/time changed Brownian motions (see Doob (1949)). This class of processes was used in Durbin and Williams (1992) to explicitly relate the FPT distribution of $X_t$ to that of the Brownian motion. Such processes fall into the larger class of diffusions which can be transformed into Brownian motion. This transformation can be done by reducing the Kolmogorov equation for a diffusion to the backward equation for Brownian motion as done in Ricciardi (1976). This article resumes an early work by Cherkasov (1957) and states alternative conditions under which a diffusion process can be transformed to a Brownian motion. Thiese types of transformations belong to a general class considered successively by Cherkasov (1980), who introduced a notion of equivalence between diffusion processes. Finally, the space/time change approach is also applicable to the inverse problem and will be used in Chapter 5, Section 5.3.

The measure change approach can be applied via the Girsanov theorem in the following way. Let $Z_t$ be a uniformly integrable positive martingale with $\mathbb{E}(Z_t) = 1$. Introduce the new probability measure $\tilde{\mathbb{P}}$ on $(\Omega, F)$ by $\tilde{\mathbb{P}}(A) = \mathbb{E}(1(A)Z_t)$ where $1(.)$ is the indicator function. With respect to the new measure $\tilde{\mathbb{P}}$, the process $X_t$ will have some drift $a(t)$ and the process $\tilde{X}_t = X_t - a(t)$ is a martingale by Girsanov’s theorem (note that this approach is also applicable to stochastic processes $a(t)$). Choosing $Z$ appropriately can transform the non-linear boundary into a linear one for the process $\tilde{X}_t$. This approach was used in e.g. Novikov (1981) to obtain lower and upper bounds for the probability $\mathbb{P}(\tau > t)$ by choosing $A = \{\tau > t\}$. Alternatively, we can use this approach to form integral equations when $a(t)$ is a linear function and $X_t = W_t$ (see Chapter 2, Section 2.2). In this case we incorporate
the linear part of the boundary into the process \( \tilde{X}_t \) for which the Radon-Nykodim derivative is computable explicitly as a function of \( X_t \) and \( t \) and acts as the kernel in the resulting integral equation of Fredholm type \( \tilde{P}(A) = \mathbb{E}(1(A)Z_t) \) for \( A := \{ \tau \leq \infty \} \).

The optional sampling theorem is the most useful tool in obtaining integral equations of Volterra (see Volterra (1930)) or Fredholm type (see Fredholm (1903)). Suppose \( Z_t = g(X_t, t) \) is a martingale then so is \( Z_{t \wedge \tau} \). Thus, after applying the optional sampling theorem to the stopping time in (1.1) and the process \( Z \), we obtain \( \mathbb{E}(Z_{t \wedge \tau}) = \mathbb{E}(Z_0) \). The last equality becomes a Volterra integral equation of the first kind after using the identity \( X_\tau = b(\tau) \) (see Chapter 2, Section 2.1), provided that \( \mathbb{E}(Z_1(\tau > t)) = 0 \). As a result the FPT problems are translated into finding appropriate martingales and solving the resulting integral equations. Such a construction produces linear integral equations in the setting of the FPT problem and non-linear integral equations for the IFPT problem. This is the approach used to obtain many of the results of this thesis.

1.1 FPT

The FPT problem has a long history starting with Bachelier (1900) who was examining the first passage of the Brownian motion to the constant boundary. His work was expanded by Paul Lévy to general linear boundary. General diffusion problems of this nature first received attention with the work of A. Khinchine, A.N. Kolmogorov and I.G. Petrovsky. Foundations of the general theory of Markov processes were laid down by Kolmogoroff (1931). This was the work which clarified the deep connection between probability theory and mathematical analysis and initiated the PDE approach to the FPT problem. For example, Khinchine (1933) looked at double constant barrier first-passage times for Markov processes and derived the solution to the 'Gambler’s ruin’ problem for Brownian motion. Khinchine also worked on deriving the PDE’s associated with the 2-dimensional version of the double barrier FPT
problems; a continuation of the work of Petrovsky (see e.g. Petrowsky (1934)).

The FPT problem for the square root boundary, for instance, was of special interest in the 1950’s and 1960’s because of its relation to the asymptotic distribution of the Anderson-Darling statistic (see Chapter 2, Section 2.2, for our derivation of the square root boundary FPT distribution). As far as the first passage time problem is concerned, the available closed form results appear to be sparse, fragmentary and essentially confined to the Brownian motion process. Hence, one is led to the study of other aspects of the FPT problem such as the asymptotic behavior of the FPT distribution and its moments (e.g. Novikov (1981), Uchiyama (1980), Peskir (2002a), Ferebee (1983)) or to setting up of ad hoc numerical procedures yielding approximate evaluations. Such procedures are either based on probabilistic approaches (e.g. Durbin (1971), Durbin and Williams (1992)) or purely numerical methods (Park and Paranjape (1974), Smith (1972), Park and Schuurmann (1976)).

In this section we outline a number of the existing developments more relevant to our results. Some of these developments, such as the results on the integral equations of Section 1.1.2 below, are directly related to our work while others have been included for completeness. In the following discussion we will assume that $X_t = W_t$ (unless otherwise specified) and $b(t)$ is a continuous function. We are interested in the cumulative distribution function of $\tau$, $F(t)$, or the density function, $f(t) := dF/dt$.

### 1.1.1 Upper and Lower Boundaries

The first (and obvious) question to ask is what is $\mathbb{P}(\tau > 0)$? That is, if $\mathbb{P}(\tau > 0) = 0$ then the Brownian motion hits the boundary instantaneously and the FPT problem is trivial. Is it possible to have $0 < \mathbb{P}(\tau > 0) < 1$? The answer to these questions is revealed by application of Blumenthal’s 0-1 law. Since $\{\tau > 0\} \in \bigcap_{s>0} F_s$, this immediately implies that $P(\tau > 0)$ is either 0 or 1. Thus, a continuous function $b : \mathbb{R}_+ \rightarrow \mathbb{R}$ is said to be a lower boundary function for $W$ if $\mathbb{P}(\tau > 0) = 1$ (otherwise $b$ is said to be an upper boundary function for $W$).
Kolmogorov’s test (see Ito and McKean (1965) p. 33-35) gives sufficient conditions which identify upper/lower functions. Recall that Kolmogorov’s test states that if \( b \) is continuous, decreasing and \( b(s)/\sqrt{s} \) is increasing then \( b \) is a lower boundary function for \( W \) if and only if:

\[
- \int_0^\infty \frac{b(s)}{s^{3/2}} \phi(b(s)/\sqrt{s}) ds < \infty
\]

(1.5)

where \( \phi \) is the standard normal density. Peskir (2002a) shows that the if-part in Kolmogorov’s test can be replaced by the statement: *If \( b \) is a continuous decreasing function satisfying \( b(s)/\sqrt{s} \to -\infty \) as \( s \downarrow 0 \), then \( b \) is a lower function for \( W \) whenever (1.5) holds.* Observe that \( b \) must satisfy \( b(0) \leq 0 \). It follows by Kolmogorov’s test that \( -\sqrt{2t \log \log 1/t} \) is an upper function for \( W \) and \( -\sqrt{(2 + \epsilon)t \log \log 1/t} \) is a lower function for \( W \) for every \( \epsilon > 0 \).

### 1.1.2 Integral Equations

Many of the *ad hoc* procedures, in the references mentioned earlier, are based on Volterra integral equations. Here, the works of Peskir (2002b) and Peskir and Shiryaev (2006) are of particular importance as the authors present a unifying approach to the integral equations arising in the FPT problem. Furthermore they generalise the class of Volterra equations of the first kind using simple calculus techniques. Essentially these Volterra equations are of the form \( \mathbb{E}(g(W_\tau, \tau)) = g(0, 0) \) i.e. they are an application of Doob’s equality and thus can be viewed as a form of a martingale method approach to FPT problems. Due to the importance of these equations in the following chapters, we present next a more detailed description of their derivation and usage.
For \( x < b(t) \) we have

\[
\Phi(x/\sqrt{t}) = \mathbb{P}(W_t < x) = \mathbb{E}(\mathbb{P}(W_t < x|\tau)) = \int_0^t \mathbb{P}(W_t < x|\tau = s)f(s)ds \quad (1.6)
\]

\[
= \int_0^t \mathbb{P}(W_{t-s} < x - b(s))f(s)ds \quad (1.7)
\]

\[
= \int_0^t \Phi((x - b(s))/\sqrt{t - s})f(s)ds \quad (1.8)
\]

where we have used \( \{W_t < x < b(t)\} \subset \{\tau \leq t\} \) together with \( W_\tau = b(\tau) \) and the independence of increments of the Brownian motion. Sending \( x \) to \( b(t) \) from below and using the dominated convergence theorem, we obtain the first Volterra integral equation of the first kind

\[
\Phi(b(t)/\sqrt{t}) = \int_0^t \Phi((b(t) - b(s))/\sqrt{t - s})f(s)ds \quad (1.9)
\]

Equation (1.9) was used in Park and Schuurmann (1976) as a basis for numerical computation of the unknown density \( f \) using the idea of Volterra to discretize the equation and solve the resulting system. This equation is especially attractive for numerical computations of \( f \) when \( b \) is given since the kernel \( K(t, s) := \Phi((b(t) - b(s))/\sqrt{t - s}) \) is nonsingular in the sense that it is bounded for all \( 0 \leq s < t \). When \( b(t) = c \) then (1.9) reads \( \mathbb{P}(\tau \leq t) = 2\Phi(c/\sqrt{t}) \) which is the reflection principle for Brownian motion. For \( b(t) = c + at, \ c < 0, \ a \in \mathbb{R} \) the equation reads

\[
\Phi((c + at)/\sqrt{t}) = \int_0^t \Phi(a\sqrt{t - s})f(s)ds
\]

which is of a convolution type. The standard Laplace transform techniques yield the following
explicit result for the density function $f$ and the corresponding c.d.f. $F$:

$$f(t) = \frac{c}{t^{3/2}} \phi\left(\frac{c + at}{\sqrt{t}}\right)$$ (1.10)

$$F(t) = \Phi\left(\frac{at + c}{\sqrt{t}}\right) + e^{-2ac} \Phi\left(\frac{-at + c}{\sqrt{t}}\right)$$ (1.11)

In Chapter 2, Section 2.2, a much simpler derivation of (1.10) will be presented.

Under the assumption that $b(.)$ is continuously differentiable on $(0, \infty)$, after differentiating (1.9) w.r.t. $t$ we obtain the following Volterra integral equation of the second kind

$$\frac{d}{dt} \Phi\left(b(t)/\sqrt{t}\right) = \frac{f(t)}{2} + \int_0^t \frac{d}{ds} \Phi\left((b(t) - b(s))/\sqrt{t - s}\right) f(s) ds$$ (1.12)

where the term $f(t)/2$ comes out of the integral since $(b(t) - b(s))/\sqrt{t - s} \to 0$ as $s \uparrow t$.

This equation was derived in Peskir (2002b) who uses it to show that when $b$ is continuously differentiable then $f$ is continuous. In the proof of the last claim the author also shows

$$\int_0^t \frac{1}{\sqrt{t - s}} f(s) ds < \infty$$ (1.13)

a result which will be used in Chapter 2. Equation 1.12 was also derived independently by Ferebee (1982) and Durbin (1985) who use probabilistic arguments. Ferebee (1983) uses this equation to obtain an expansion for $f$ in terms of the Hermite polynomials. Durbin and Williams (1992) provide yet another derivation of this equation.

Integration by parts applied to (1.9) produces another Volterra equation of the second kind

$$\Phi\left(b(t)/\sqrt{t}\right) = \frac{F(t)}{2} - \int_0^t \frac{d}{ds} \Phi\left((b(t) - b(s))/\sqrt{t - s}\right) F(s) ds$$ (1.14)

This equation was used in Peskir (2002b), for the class of boundaries $b(t) \geq b(0) - c \sqrt{t}, c >$
0, \( b(0) < 0 \) of continuously differentiable, decreasing convex functions, to prove uniqueness of a solution to (1.9) and to derive its infinite series representation. The uniqueness result follows from the fixed point principle for contractive mappings on a complete metric space. The author shows that the mapping

\[ T : G \rightarrow 2\Phi(b(t)/\sqrt{t}) + \int_0^t \frac{d}{ds}\Phi((b(t) - b(s))/\sqrt{t-s})G(s)ds, \]

on the Banach space \( B(\mathbb{R}_+) \) of all bounded functions \( G : \mathbb{R}_+ \rightarrow \mathbb{R} \) equipped with the sup norm, is a contraction from \( B(\mathbb{R}_+) \) into itself implying the uniqueness. The same equation, (1.14), was derived in Park and Paranjape (1974) using the same technique. They also use the fixed point principle on the Hilbert space \( L^2 \) to show uniqueness of a solution to (1.9) for continuously differentiable boundaries that satisfy \( |b'(t)| \leq c/t^p, \ p < 1/2 \) and give its infinite series representation. In Chapter 2, we obtain a similar sufficient condition for the existence of a unique solution to a class of new integral equations. The results of Park and Paranjape (1974) were generalised by Ricciardi et al. (1984) to the class of diffusion processes. Furthermore, for the case of Brownian motion, they expand the result on the uniqueness of an \( L^2 \) solution to (1.9) for boundaries of the form \( b(t) = a + ct^{1/p}, \ a, b, p \in \mathbb{R}, \ p > 2 \).

Going back to equation (1.8), differentiating it w.r.t. \( x \) and taking the limit \( x \uparrow b(t) \), we obtain the second Volterra equation of the first kind

\[
\frac{1}{\sqrt{t}}\phi(b(t)/\sqrt{t}) = \int_0^t \frac{1}{\sqrt{t-s}}\phi\left(\frac{b(t) - b(s)}{\sqrt{t-s}}\right)f(s)ds
\]

(1.15)

This equation was derived in Durbin (1971) where it is used to obtain a numerical solution for \( f \) by approximating the boundary by straight line segments on subintervals \((s, s + ds)\) and using available results for crossing probabilities for linear boundaries. Subsequently, Smith (1972) recognizes (1.15) as a Generalized Abel equation and proposes Abel’s linear
transformation \( T: g \rightarrow \int_0^y g(t)/\sqrt{y-t}dt \), to deal with the singularity of the kernel at \( s = t \). He then solves the equation numerically using standard quadrature methods.

Multiplying (1.15) by \( b'(t) \) (assuming \( b \) is differentiable) and adding it to (1.12) we get

\[
\frac{b(t)}{t^{3/2}} \phi \left( \frac{b(t)}{\sqrt{t}} \right) = f(t) + \int_0^t \frac{b(t) - b(s)}{(t-s)^{3/2}} \phi \left( \frac{b(t) - b(s)}{\sqrt{t-s}} \right) f(s) ds
\]  

(1.16)

This equation was derived and studied by Ricciardi et al. (1984) using other means. Furthermore, the authors use it to generalise the results of Park and Paranjape (1974).

Going back to equation (1.8), the standard rule of differentiation under the integral sign produces the class of integral equations

\[
\frac{1}{t^{(n+1)/2}} \frac{d^n}{dx^n} \phi \left( \frac{x}{\sqrt{t}} \right) = \int_0^t \frac{1}{(t-s)^{(n+1)/2}} \frac{d^n}{dx^n} \phi \left( \frac{x-b(s)}{\sqrt{t-s}} \right) f(s) ds
\]  

(1.17)

for all \( x < b(t) \) and \( n \geq 0 \). Recall that \( \frac{d^n}{dx^n} \phi(x) = (-1)^n H_n(x) \phi(x) \) where \( H_n \) are the Hermite polynomials of order \( n \). This class of equations was obtained by Peskir (2002b) and, in Chapter 2, we will show an alternative derivation using our martingale approach.

As mentioned at the beginning of the section, a unified approach in the derivation of the Volterra equations arising from the FPT framework was presented in Peskir (2002b). The author, also, generalises the class of Volterra equations of the first kind by the following result:

**Theorem 1** Let \((W_t)_{t \geq 0}\) be a standard Brownian motion and \( \tau \) be the first passage time of \( W \) over the continuous boundary \( b: (0, \infty) \rightarrow \mathbb{R} \). Let \( f \) denote the density of \( \tau \). Then the following system of integral equations is satisfied:

\[
t^{n/2} G_n \left( \frac{-b(t)}{\sqrt{t}} \right) = \int_0^t (t-s)^{n/2} G_n \left( \frac{-b(t) - b(s)}{\sqrt{t-s}} \right) f(s) ds
\]

(1.18)
for $t > 0$ and $n = -1, 0, 1, \ldots$, where we set

$$G_n(x) = \int_x^\infty G_{n-1}(z)dz$$  \hspace{1cm} (1.19)$$

with $G_{-1} = \phi$ being the standard normal density.

For completion we outline the author’s proof of this result. Integrate (1.8) (as a function of $x$) on $(-\infty, z)$, $z < b(t)$, and make the substitution $u = x/\sqrt{t}$ and $v = (x - b(s)/\sqrt{t - s})$. Exchanging the order of integration we obtain

$$\sqrt{t} \int_{-\infty}^{z/\sqrt{t}} \Phi(u)du = \int_0^t \sqrt{t - s} \int_{-\infty}^{(z-b(s))/\sqrt{t-s}} \Phi(v)dv f(s)ds$$

The last equation is the same as (1.18) with $n = 1$. Proceeding in this manner, the result follows by induction. Note that no equation of the system (1.18) is equivalent to another equation from the same system except for itself. We will generalise this class of Volterra equations in Section 2.1.

1.1.3 Small Time Behavior of the FPT Density

Many of the numerical procedures mentioned at the beginning of this section are based on numerical solutions of the integral equations discussed above. For this reason, knowledge of the behavior of the density function near 0 is essential for the construction of a numerical solution since it provides an informed first estimate or starting point in a numerical algorithm. In this section we outline some important results on this topic, presented in Peskir (2002a).

From equation (1.9), which holds for any continuous function $g$ satisfying $g(0) \geq 0$ whenever the limit $f(0)$ exists (and is finite), we can obtain the following formula:

$$f(0) = \lim_{\epsilon \to 0} \frac{\Phi(b(t)/\sqrt{t})}{\int_0^t \Phi(b(t)/\sqrt{t-s})ds}$$
Although the last result looks attractive, it is difficult to see how to compute this limit. A similar argument applied to equation (1.15) leads to yet another explicit formula for $f(0)$. A more promising approach is to use (1.16) under the assumption that $b$ is continuously differentiable and decreasing (locally at 0). If the following condition is satisfied

$$\frac{b(t)}{t^{3/2}} \phi(b(t)/\sqrt{t}) \to 0$$

as $t \downarrow 0$, we see that we must have $f(0) = 0$. This is certainly the case when $b(0) < 0$ and $b(t)$ decreasing (locally). Peskir (2002a) completes the argument in obtaining (1.20) and proves that if $b$ is $C^1$ on $(0, \infty)$, decreasing (locally) and convex (locally) then

$$f(0) = \lim_{t \downarrow 0} \frac{b(t)}{2t^{3/2}} \phi(b(t)/\sqrt{t}) = \lim_{t \downarrow 0} \frac{b'(t)}{\sqrt{t}} \phi(b(t)/\sqrt{t})$$

whenever the second and third limits exist. As mentioned above, the convexity restriction on $b$ is unnecessary if the first limit is 0. The author also relaxes the convexity assumption to show that $f(0) = 0$ whenever $b(0) < 0$ and $b(t)$ is either decreasing (locally) or increasing (locally). Here is the reasoning behind this result: Consider two boundaries $b_i$, $i = 1, 2$ which satisfy $b_1(t) \leq b_2(t)$ for all $t \in (0, \delta)$, $\delta > 0$. Let $f_1$ and $f_2$ denote the corresponding densities of the resulting FPT’s $\tau_1$ and $\tau_2$, with limits $f_1(0)$ and $f_2(0)$. Since $b_1 \leq b_2$ we have $P(\tau_1 < t) \leq P(\tau_2 < t)$ for all $t \in (0, \delta)$ and thus $P(\tau_1 < t)/t \leq P(\tau_2 < t)/t$. Passing to the limit as $t \downarrow 0$ we get $f_1(0) \leq f_2(0)$. This is the comparison principle for FPT densities. With this in mind we see that whenever $b(t) \leq -\sqrt{(2 + \epsilon)t \log(1/t)} =: b_\epsilon(t)$, for all $t \in (0, \delta)$ and some positive $\epsilon$ and $\delta$, then $f(0) = 0$. This follows from (1.21) applied to $b_\epsilon$ together with the comparison principle. Similar argument shows that if $b(t) \geq \sqrt{2t \log(1/t)}$ then $f(0) = \infty$. These results are proven and discussed in more detail in Peskir (2002a).
1.4 PDE Approach

Among the closed form results related to the FPT problem, one of the more attractive constructions for explicit computation of the FPT density was presented by Lerche (1986). The author applies the PDE approach to solve the system (1.2). More formally, let us define the function:

$$h(x,t) = \frac{1}{\sqrt{t}} \phi(x/\sqrt{t}) - \frac{1}{a} \int_{0}^{\infty} \frac{1}{\sqrt{t}} \phi((x + \theta)/\sqrt{t})Q(d\theta), \ a > 0 \quad (1.22)$$

where $Q$ is a $\sigma$-finite measure satisfying $\int_{0}^{\infty} \phi(\sqrt{\epsilon} \theta)Q(d\theta) < \infty$, $\forall \epsilon > 0$. The equation $h(x,t) = 0$, for a fixed $t$, is equivalent to the equation

$$h_0(x,t) := \int_{0}^{\infty} e^{-\theta x/t - \theta^2/2}Q(d\theta) = a \quad (1.23)$$

and thus, for each fixed $t > 0$, has a unique convex solution, denoted by $b_a(t)$. The uniqueness follows from the monotonicity of $h_0$, as a function of $x$, and the convexity follows from Holder’s inequality applied to (1.23). Furthermore, Lerche (1986) shows that $b_a(t)$ is infinitely often continuously differentiable lower boundary, $b_a(t)/t$ is monotone increasing and $h(x,t)$ satisfies the diffusion equation (1.2) with $\mu = 0, \sigma = 1$. Thus, the probability density function $f_a(t)$, of the FPT of the Brownian motion to the boundary $b_a$, is given by:

$$f_a(t) = \frac{1}{2} h_x|_{x=b_a(t)} = \frac{\phi(b_a(t)/\sqrt{t}) \int_{0}^{\infty} \theta \phi(b_a(t)/\sqrt{t})Q(d\theta)}{2t^{3/2} \int_{0}^{\infty} \phi(b_a(t)/\sqrt{t})Q(d\theta)} \quad (1.24)$$

The main drawback of this construction is that we do not start with the boundary $b$ but with the measure $Q$ which, in turn, defines $b$. Hence, in most cases $b$ is an implicit function and then so is $f$. Some of the more interesting explicit examples presented in Lerche (1986)
are the triples \((Q, b_a, f_a)\) given by

\[
(\delta(2\theta), \theta - \log(a)/(2\theta)t, \theta \phi(b_a(t))/t^{3/2})
\]

and

\[
\left(\frac{d\theta}{\sqrt{2\pi}}, -\sqrt{t \log(a^2/t)}, \frac{b_a(t)}{2t^{3/2}} \phi(b_a(t)/\sqrt{t})\right), \ t \leq a^2
\]

Durbin (1985) obtains the following explicit formula for the FPT density of a continuous Gaussian process \(X_t\) to a boundary \(b(t)\):

\[
f(t) = g(t)k(t) \tag{1.25}
\]

where \(k(t)\) is the density of the process on the boundary and \(g(t)\) is given by

\[
g(t) = \lim_{s \uparrow t} (t - s)^{-1} \mathbb{E}(1(s, t)(X_s - b(s)) | X_t = b(t)) \tag{1.26}
\]

with \(1(s, t)\) denoting the indicator function defined to equal 1 if the sample path does not cross the boundary prior to time \(s\) and 0 otherwise. On first glance it does not appear that expressions (1.25) and (1.24) give the same result in the case \(X_t = W_t\). However, Durbin (1988) demonstrates the equivalence of the two formulas for the Brownian motion and boundaries defined in the Lerche (1986) set-up.

### 1.2 Inverse FPT

The problem of finding the boundary \(b\) given a distribution function for \(\tau\) was first posed by A. Shiryaev in 1976 and is of critical importance in many problems related to modern credit risk management. The IFPT setting is a natural approach to model default time of a company. Hull and White (2000) and Hull and White (2001) show how the required input
in the inverse problem, the (risk-neutral) density $f$, can be extracted from observed market prices. Furthermore, they show how the resulting boundary can be used, once computed, in a model for pricing credit default swaps with counterparty default and remark that it could be used to price other, exotic, credit derivatives. Iscoe et al. (1999) show how the inverse passage problem is a key component in a multistep integrated market and credit risk portfolio model. The IFPT problem is much more challenging than the FPT problem and there has been relatively little work done on it. Two early papers written by Dudley and Gutmann (1977) and Anulova (1980) deal with the existence of some stopping times for a given distribution, however, these stopping times are not of the form (1.1) for some function $b$. Most of the work up to date is concerned with the numerical calculation of the boundary for a given density. Avellaneda and Zhu (2001) applied the finite difference scheme to solve a PDE formulation of the IFPT problem. Zucca et al. (2003) applied the secant method to the integral equations of Peskir (2002b). Iscoe and Kreinin (2002) demonstrated that a Monte-Carlo approach can be applied to solve the inverse problem in discrete time, by reducing it to the sequential estimation of conditional distributions. Hull and White (2001) also considered a time discretization and computed the boundary by solving a system of nonlinear equations at each time point. One of the main contributions to the description of the inverse problem thus far, was provided by the work of Chadam et al. (2006a) where the authors show the existence of a unique viscosity weak solution for the IFPT problem. Some of their results will be discussed next. Formulation of the IFPT problem in a PDE setting is done as follows. Define the function $w(x, t) := \mathbb{P}(W_t > x, \tau > t)$.

From the Kolmogorov forward equation, $w(x, t)$ satisfies the following free boundary
problem

\[ Lw = 0 \text{ when } w(., t) < p(t) \]  \hspace{1cm} (1.27)
\[ 0 \leq w(x, t) \leq p(t) \text{ for any } (x, t) \in (\mathbb{R} \times (0, \infty)) \]  \hspace{1cm} (1.28)
\[ w(x, 0) = 1(-\infty, 0) \text{ for } x \in \mathbb{R} \]  \hspace{1cm} (1.29)

where \( Lw := w_t - (1/2)w_{xx} \) and \( p(t) := P(\tau > t) \). Thus, \( x = b(t) \) is the solution to \( w(x, t) = p(t) \). Chadam et al. (2006a) show the existence of a viscosity solution to (1.27). Furthermore, the authors provide upper and lower bounds on the asymptotic behavior of the boundary and obtain the following small time behavior of the boundary:

\[ \lim_{t \downarrow 0} \frac{b(t)}{\sqrt{-2t \log(F(t))}} = -1 \]  \hspace{1cm} (1.30)

provided that \( \limsup_{t \downarrow 0} \frac{F(t)}{tf(t)} < \infty \). Finally, the authors derive equations (1.9), (1.15) and (1.16) and show that \( b \) is the solution to the free boundary problem (assuming continuous \( p \)), provided that one of the following holds

- \( b \) satisfies (1.9) for all \( t \in (0, T] \).
- \( b \) satisfies (1.15) for all \( t \in (0, T] \) and the function \( q(t) := \int_0^t f(s) / \sqrt{2\pi(t-s)} ds \) is continuous on \((0, T]\) with \( q(0) = 0 \).
- \( b \) satisfies (1.16), \( \lim_{t \downarrow 0} b(t) / \sqrt{t} = -\infty \), and the function \( q^1(t) := \int_0^t \frac{|b(t) - b(s)|}{(t-s)^{3/2}} f(s) ds \) is continuous and uniformly bounded on \((0, T]\).

Note that in the context of the integral equations discussed so far, the FPT problem seeks a solution to a linear integral equation, while the IFPT problem is equivalent to finding a solution to a nonlinear integral equation and such equations are known to exhibit non-unique solutions. In light of this, the results of Chadam et al. (2006a) are of particular importance and obtaining them through the integral equations approach may be a formidable task.
1.3 Main Results and Outline

As we demonstrated thus far, obtaining any analytical solution to the first passage time problems is a difficult task even for a small class of boundaries. On the other hand there exist, already, a large number of numerical procedures in the current literature most of which are based on an integral equation of Volterra type. For these reasons, in Chapter 2, we have concentrated on the development of new tools for the classical first passage time. The focus there is on the unification of the construction of existing integral equations and the development and examination of new equations. The two types of integral equations analyzed are Volterra and Fredholm equations of the first kind and in Chapter 2 we show the connection between the construction of such integral equations and the theory of martingales.

The construction of Volterra equations is based on the identification of martingales with certain properties and the application of the optional sampling theorem which produces the actual equation. In order to motivate the construction procedure we will take the integral equation (1.8).

\[
\Phi \left( \frac{x}{\sqrt{t}} \right) = \int_0^t \Phi \left( \frac{x - b(s)}{\sqrt{t-s}} \right) f(s) ds
\]

where \( f \) is the density function of the the first passage time and \( x < b(t) \). This equation can be written in the form

\[
\mathbb{E}(X_\tau 1(\tau \leq t)) = X_0
\]  

(1.31)

where the process \( X_\tau \) is defined as \( X_\tau = \Phi \left( \frac{x-W}{\sqrt{t-s}} \right) \). Since the function \( u(x,t) := \Phi(x/\sqrt{t}) \) satisfies the partial differential equation \( u_t = u_{xx}/2 \), it follows, by applying Ito’s lemma, that \( X_\tau \) has a zero drift and thus is a real-valued martingale for \( s < t \). The above form of the equation is almost the same as the result of the optional sampling theorem with the difference being the indicator function inside the expectation. However, it is not difficult to see that \( X_\tau 1(\tau > t) = 0 \) a.s.. The reasoning behind this result is that on the set \( \{\tau > t\} \) we
have $W_t > b(t) > x$. Therefore \( \lim_{s \uparrow t} \frac{W_t}{\sqrt{t-s}} = -\infty \) on the set \( \{ \tau > t \} \). It follows that (1.31) can be written as

\[
\mathbb{E}(X_\tau) = X_0
\]

which is precisely the result of the optional sampling theorem. The final step is to pass to the limit $x \uparrow b(t)$ and to examine sufficient conditions for the resulting equation to have a unique solution. Thus, the first step in constructing Volterra integral equation of the first kind is to identify martingales, which, for arbitrary $t > 0$ posses the property $X_{\tau,1}(\tau > t) = 0$. It turns out that such martingales can be constructed using the general solution to the heat equation on an infinite rod as given in Widder (1944). This construction is discussed in Section 2.1. In Section 2.1.1 we examine the passage to the limit $x \uparrow b(t)$ for a particular class of integral equations. This class contains all currently known Volterra equations of the first kind arising from the FPT problem which were listed in the previous section. In Section 2.1.2 we prove this fact. Section 2.1.3 is dedicated to obtaining sufficient conditions under which these equations exhibit a unique continuous solution. Functional transforms of the boundary and the corresponding first passage times are discussed in Section 2.1.4. The rest of Chapter 2 is focused on obtaining Fredholm equations of the first kind using a simple measure change technique. These simple equations alone allow us to derive the analytical results for the three classes of boundaries; linear, square-root and quadratic. In other words, for these three classes, we present an alternative derivation of the existing results and thus unify the three best known cases in the FPT problem. At the end of Chapter 2 we examine the connection between the Fredholm equations and the class of Volterra equations.

In Chapter 3 we examine a modification of the classical FPT problem which we call the randomized FPT or the matching distribution (MD) problem. Under this problem the object of interest is the randomized first passage time

\[
\tau_X := \inf\{t > 0; W_t + X \leq b(t)\}
\]
where $X$ is a random variable independent of the Brownian path. Furthermore, $X$ can be viewed as the random starting point of the Brownian motion. This second source of randomness provides flexibility and allows us to assume not only the boundary $b(t)$ as an input but also the (unconditional) distribution of $\tau_X$, $f_{\tau_X}$. The MD problem seeks the distribution of $X$ which matches the pair $(b, f)$. This framework is not only applicable to 'real life' settings (such as mortality of a cohort) but it also produces a partial solution to both the FPT and IFPT problems since both the boundary function and the distribution function of $\tau_X$ are assumed as given. More formally, the problem seeks a solution to the equation

$$\mathbb{E}_X(f(t|X)) = f_{\tau_X}(t)$$

where $f(t|X)$ is the conditional distribution of $\tau$ given $X$ and the expectation is taken under the law of $X$. Section 3.1 deals with the general boundary case. We show sufficient conditions for the existence of a random variable $X$ and we derive the Laplace and Hermite transforms of the density function of $X$ (assuming it exists) using the Volterra and Fredholm equations obtained in Chapter 2. These two transforms provide us with semi-analytical solution to the MD problem and show that if a solution exists it is unique. Furthermore, we address the relationship between different boundaries and their corresponding matching distributions. In Section 3.2 we discuss the case of the linear boundary end explore the Laplace transform of the matching distribution. For the linear boundary the random starting point of the Brownian motion is essentially the random intercept of the linear boundary. We obtain analytical results for the matching distribution under a large class of unconditional distributions (which is an infinite mixture of gamma distributions) for $\tau$. Finally, we discuss the case of a random slope and its relationship to the random intercept. In Section 3.3 we motivate the use of the MD problem and the results obtained for it to attack the classical FPT problem. We derive integral equations in the FPT setting involving new quantities and use the randomization technique to obtain known transforms of these quantities.
In Chapter 4 we apply the randomization technique of Chapter 3 to model the mortality of a Swedish cohort. In the model, individuals are assumed to possess a number of 'health' units given by the health process

\[ h_t = X - t + \beta W_t \]

where \( X \) is the random amount of 'health' at birth and \( W_t \) is a standard Brownian motion representing the individual fluctuations in the health process. Individuals are assumed to die at time

\[ \tau := \inf\{t : h_t = 0\}. \]

Based on the mortality data we can fit a mixture of gamma distribution to approximate the distribution of \( \tau \) with desired precision, which in turn gives us the distribution of \( X \) using the results of Chapter 3. However, certain restrictions apply under which there is a trade-off between the level of precision in the fit and the maximum amount of volatility allowed given by \( \beta \). Nevertheless, the model provides a dynamical and structural setting particularly suited for pricing mortality linked financial products. For this purpose the 'risk-neutral' measure is induced by a slope change in the process \( h_t \) which under the objective measure is set at \(-1\). Therefore the transition to a 'risk-neutral' measure has a natural interpretation since the slope of \( h_t \) represents the average rate of decrease of the health process. By changing the slope we simply express our belief that individuals die faster/slower under the 'risk-neutral' measure.

In Chapter 5 we discuss some applications of the integral equations of Chapter 2. The chapter starts with an application of the method of Lerche (1986) and equation (1.24) to a class of boundaries which, to our knowledge, have not been discussed in the context of FPT thus far. This class of boundaries is the solution to equation (1.23) for the particular \( \sigma \)-finite measure \( Q(d\theta) = \theta^{p-1}d\theta, p > 0 \). This measure plays a fundamental role in deriving the class
of Volterra integral equations of Chapter 2. For this reason we suspect that the results of Lerche (1986) can be obtained from the Volterra equations showing the equivalence of the two approaches. However, this is a topic for future research and we do not explore it further in this work.

Section 5.2 deals with the use of the Fredholm equation of Chapter 2 to obtain the coefficients in the expansion of the FPT density with respect to the Laguerre polynomials which form a complete orthogonal basis in $L^2$ with respect to the standard exponential distribution. The Fredholm equation is particularly suited for these orthogonal polynomials because of the exponential form of its kernel. It provides us with a linear system, the solution to which is the set of coefficients in the expansion of the FPT density. However, due to the unstable nature of the system we employ the regularization method of Tikhonov (1963) to deal with the ill-posedness of the problem. A number of examples are presented where we compare the numerical results based on the Laguerre polynomials expansion and the numerical solution to one of the Volterra integral equations for a few boundaries.

In Section 5.3 we investigate space/time changed Brownian motions. Whenever the space change is monotone we can reverse the space and/or time change and reduce these processes to a standard Brownian motion and thus all results obtained for the Brownian motion apply to these processes as well. One particular application is for the IFPT problem. If the distribution of $\tau$ is in the scale family of distributions we show that the corresponding boundary would satisfy the same scaling property. In particular if $\tau \sim F_\lambda$ where $F_\lambda$ is in the scale family of distributions then the corresponding boundary can be written as $b_\lambda(t) = b_1(\lambda t)/\sqrt{\lambda}$. This result shows that if we know the boundary corresponding to the distribution with parameter $\lambda = 1$ then we can obtain the boundary corresponding to a general parameter $\lambda$. Thus, in the case of the scale family of distributions, the IFPT problem is reducible to finding a single boundary called the base boundary. This idea is applied to the exponential $Exp(\lambda)$ and uniform $U([0, \lambda])$ distributions by finding the base boundary.
(for $\lambda = 1$) using a numerical solution to one of the Volterra integral equations, fitting a functional form for the boundary and obtaining the functional form for the boundary corresponding to a general $\lambda$. 
Chapter 2

Integral Equations

2.1 Volterra Integral Equations

The Volterra equations of the first kind listed in the previous chapter can all be written in the form

\[ E(m(W_\tau, \tau)) = m(0, 0) \]

where \( m(W_s, s) \) is a martingale and \( \tau := \inf\{t > 0; W_t \leq b(t)\} \) is the first passage time of the Brownian motion \( W_t \) to the regular boundary \( b(t) \). These equations can be viewed as a product of the optional sampling theorem applied to an appropriate martingale \( m(W_s, s) \).

In this section, using this simple martingale result, our main aim is to present a unifying approach to the Volterra integral equations arising from the FPT and generalise the known class of integral equations. We start with a motivating example connecting a known Volterra equation to a certain martingale and show how to use martingales with certain properties to construct more general Volterra equations. We then introduce a class of functions which generate such martingales and derive a corresponding class of integral equations which embed the currently known Volterra equations of the first kind.

We assume that \( b(t) \) is a regular boundary in the sense that \( \mathbb{P}(\tau = 0) = 0 \). Sufficient
conditions for regularity are given by Kolmogorov’s test (see e.g. Ito and McKean (1965) pp. 33-35). Furthermore, we allow \( b(0) = -\infty \) but we assume that whenever this is the case then there exists \( \epsilon > 0 \) such that \( b \) is monotone increasing on \((0, \epsilon]\). Thus we define the following class of boundary functions

**Definition 1** Let \( \mathcal{D} \) denote the class of regular boundary functions \( b : [0, \infty) \to \mathbb{R} \cup \{-\infty\} \), continuous on \((0, \infty)\), and for which if \( \lim_{t \downarrow 0} b(t) = -\infty \) then there exists \( \epsilon > 0 \) such that \( b \) is monotone increasing on \((0, \epsilon]\).

The motivation behind the attempt to connect the theory of martingales and the construction of integral equations for Brownian motion is perhaps best illustrated by the following well known Volterra equation mentioned in the previous chapter:

\[
\int_0^t \phi \left( \frac{y - b(s)}{\sqrt{t-s}} \right) / \sqrt{t-s} F(ds) = \phi(y/\sqrt{t})/\sqrt{t}
\]

where \( \phi \) is the standard normal density function. The equality holds for all \( y \leq b(t) \) for continuous regular boundaries \( b \). This equation can be written as

\[
\mathbb{E}(X_\tau 1(\tau \leq t)) = X_0
\]

where the process \( X_s \) is defined as \( X_s = \phi \left( \frac{y-W_s}{\sqrt{t-s}} \right) \) for fixed \( t > 0 \). Noting that \( X_s \) is a real-valued martingale for \( s < t \) and that \( X_t 1(\tau > t) = 0 \) a.s. (since on the set \( \{ \tau > t \} \) we have \( W_t > b(t) > y \)), equation (2.1) can be viewed as a product of the optional sampling theorem applied to the process \( X_{s\wedge t} \) and the stopping time \( \tau \). The following Proposition outlines the construction of Volterra equations of the form (2.1).

**Proposition 1** Let \( m(x,s), \ s < t \) be a real-valued function such that \( X_s := m(W_s,s) \) is a martingale satisfying

1) \( \mathbb{E}(|X_\tau 1(\tau \leq t)|) = \int_0^t |m(b(s),s)| F(ds) < \infty \)
2) \( \lim_{s \uparrow t} \mathbb{E}(X_s 1(\tau > s)) = 0 \)

Then

\[
m(0, 0) = \int_0^t m(b(u), u) F(du)
\]

**Proof.** Supposing such a martingale exists, take a localizing sequence of stopping times \( s \land \tau, \ s < t \). Then, applying the optional sampling theorem to \( X \) and \( s \land \tau \) and passing to the limit \( s \uparrow t \) and using the dominated convergence theorem, we obtain

\[
X_0 = \lim_{s \uparrow t} \mathbb{E}(X_{s \land \tau}) = \lim_{s \uparrow t} \mathbb{E}(X_\tau 1(\tau \leq s)) = \int_0^t m(b(u), u) F(du)
\]

by the use of the almost sure identity \( W_\tau = b(\tau) \). \( \square \)

Thus, the first step is to look for a class of martingales of the form \( X_s := m(W_s, s), \ s < t \), satisfying the conditions of Proposition 1. The class of functions \( m \) for which the process \( X_s \) satisfies the above properties is rather large. A subclass of positive functions \( m \) can be constructed using the following result due to Widder (1944):

**Theorem 2 (Widder (1944))** Let \( u(., .) \) be a continuous, non-negative function on \( I = (0, \delta) \times \mathbb{R}, \ 0 < \delta \leq \infty \). The following statements are equivalent:

1) \( u(s, x) \) satisfies the diffusion equation \( u_s = u_{xx}/2 \) on \( I \) and \( \lim_{(s, x) \rightarrow (0, e)} u(s, x) = 0 \) for all \( e < 0 \)

2) There exists a positive \( \sigma \)-finite measure \( Q \) on \( [0, \infty) \) such that \( u(s, x) \) can be represented as

\[
u(s, x) = \int_0^\infty 1 \frac{1}{\sqrt{s}} \phi\left(\frac{x - \theta}{\sqrt{s}}\right) Q(d\theta)
\]

(2.2)

Given this result, define \( m(x, s) := u(t - s, y - x) \) for a fixed \( t > 0 \) and \( y < b(t) \). Then \( m \) satisfies the diffusion equation \( m_s = -m_{xx}/2 \) using the first part of the above theorem. Furthermore, the process \( X_s := m(W_s, s), \ s < t \) is a martingale (we can check directly, by computing the double integral, that \( \mathbb{E}(|X_s|) = X_0 < \infty \) for all \( s < t \)). Checking the first
condition, $\mathbb{E}(|X_\tau|1(\tau \leq t)) < \infty$, we have:

$$
\int_0^t |m(b(s), s)|F(ds) = \int_0^\infty Q(d\theta) \int_0^t \frac{1}{\sqrt{t-s}} \phi\left( \frac{b(s) - (z - \theta)}{\sqrt{t-s}} \right) F(ds) = u(z, t)
$$

using the equation in (2.1). Furthermore, note that on the set $\tau > s$ we have $W_s > b(s)$. Take $s_0$ close enough to $t$ and such that for all $s_0 < s \leq t$ we have $b(s) > y$. Such $s_0$ exists since $b$ is continuous and $b(t) > y$. Then

$$
\mathbb{E}(X_s1(\tau > s)) = \int_0^\infty Q(d\theta) \mathbb{E}(1(\tau > s)\phi\left( \frac{y - W_s - \theta}{\sqrt{t-s}} \right))/\sqrt{t-s}
$$

$$
\leq \int_0^\infty \frac{1}{\sqrt{t-s}} \phi\left( \frac{y - b(s) - \theta}{\sqrt{t-s}} \right) Q(d\theta) = u(t-s, y - b(s))
$$

Taking the limit $s \uparrow t$ and using the limiting behavior of the function $u$ as given in the theorem above we see that $\lim_{s\uparrow t} \mathbb{E}(X_s1(\tau > s)) = 0$. Therefore, by Proposition 1, we obtain the Volterra equation of the first kind:

$$
u(t, y) = X_0 = \mathbb{E}(X_\tau1(\tau \leq t)) = \int_0^t u(t-s, y - b(s))F(ds) \quad (2.3)
$$

for any $y < b(t), \ t > 0$.

The integral representation of the function $u$ (equation (2.2)) is computable for several specific “degenerate” cases, such as when $Q(d\theta)$ is a sum of Dirac measures or a uniform measure over a compact domain. However, we have found one other general class of measures which lead to tractable forms for $u$ itself, specifically when $Q(d\theta) = \theta^{-p-1}d\theta$ for $p < 0$. In this case by direct calculation (see (A.4)) we have

$$
u(t, x; p) = e^{-x^2/(4t)}D_p(-x/\sqrt{t})\sqrt{t}^{-p-1}\Gamma(p)/\sqrt{2\pi}
$$

where $D_p$ is the parabolic cylinder function (see Section A.0.1). Note that for $p \geq 0$ this
particular $u(t, x; p)$ still satisfies the diffusion equation $u_t = u_{xx}/2$; furthermore $u(t, x; 0) \sim \phi(x/\sqrt{t})/\sqrt{t}$ and $u(t, x; -1) \sim \Phi(x/\sqrt{t})$ which are the kernels of the two well known Volterra equations. These observations motivate us to examine the process arising from the function

$$m(s, x; p, t) = e^{-\frac{(x-y)^2}{4(t-s)}} D_p((x-y)/\sqrt{t-s})/(t-s)^{(p+1)/2}, \ p, y, x \in \mathbb{R},$$

for a fixed $t > 0$. Define the process $X_s = m(s, W_s; p, t)$. It is important to point out that time flows with $s$, while $t$ represents a fixed time point. We now proceed to show that $X_s$, $s < t$, is an honest martingale.

Lemma 1 The process $X_s$, $s < t$ given by

$$X_s = e^{-\frac{(W_s-y)^2}{4(t-s)}} D_p((W_s-y)/\sqrt{t-s})/(t-s)^{(p+1)/2}$$

is a real valued martingale for all $p, y \in \mathbb{R}$, $t > 0$.

Proof. Using the second order differential equation (A.3), to which $D_p$ is a solution, it is straightforward to show that

$$m_s = -1/2m_{xx}$$

To check the integrability condition, consider $\mathbb{E}|m(s, W_s; p)|1(|W_s| > a), a \gg y, s < t$. Using the asymptotic behavior of the parabolic cylinder function (A.10) and (A.9) we obtain:

$$\mathbb{E}|m(s, W_s; p)|1(|W_s| > a) = \int_{-\infty}^{-a} |m(s, x; p, t)| \frac{e^{-x^2/(2s)}}{\sqrt{2\pi s}} dx + \int_{a}^{\infty} |m(s, x; p, t)| \frac{e^{-x^2/(2s)}}{\sqrt{2\pi s}} dx$$

$$\sim \int_{-\infty}^{-a} \frac{e^{-\frac{x^2}{2\pi}} |(x-y)^{-p-1}|}{\sqrt{2\pi s}} dx$$

$$+ \int_{a}^{\infty} \frac{e^{-\frac{(x-y)^2}{2(t-s)}} x^p}{(t-s)^{p+1/2} \sqrt{2\pi s}} dx < \infty$$
Furthermore, \( m(s, x; p, t) \) is a continuous function in \( x \) on \([-a, a]\). Thus

\[
E(|X_s|) = E|m(s, W_s; p, t)| = E|m(s, W_s; p, t)|1(|W_s| > a) + E|m(s, W_s; p, t)|1(|W_s| \leq a) < \infty
\]

Therefore \( X_s \) is a martingale for all \( p, y \in \mathbb{R} \). \( \square \)

For \( s < t \) the process \( X_s \) is a real valued martingale while for \( s > t \) it is a complex valued martingale. In order to construct the class Volterra equations with kernel functions \( m(s, b(s); p, t) \) we combine Proposition 1 and Lemma 1 to obtain the following

**Theorem 3** Let \((W_t)_{t \geq 0}\) be a standard Brownian motion with boundary function \( b \in \mathcal{D} \). Let \( \tau \) be the first-passage time of \( W \) below \( b \), and let \( F \) denote its distribution function. Then for all \( p \in \mathbb{R} \) and \( y < b(t) \) the following system of integral equations is satisfied:

\[
e^{-\frac{y^2}{4t}} D_p(-y/\sqrt{t}) = \int_0^t e^{-\frac{(b(s)-y)^2}{4(t-s)}} D_p((b(s)-y)/\sqrt{t-s}) \frac{1}{(t-s)^{(p+1)/2}} F(ds) \quad (2.7)
\]

where \( F \) is the distribution of \( \tau \).

**Proof.** Define the stopping time \( \tau_t = \tau \wedge t \), and fix \( y \in \mathbb{R} \) such that \( y < b(t) \) and set \( t > 0 \). As previously mentioned, on the set \( \{\tau_t > s\} \), we have \( W_s > b(s) \) which implies

\[
(W_s - y)/\sqrt{t-s} > (b(s) - y)/\sqrt{t-s} \to \infty
\]

as \( s \uparrow t \) because of the continuity of \( b(.) \) and the condition \( y < b(t) \). Thus, choosing \( s_0 \) close enough to \( t \) and such that for all \( s_0 < s \leq t \) we have \( b(s) > y \) and using the asymptotic
behavior of the parabolic cylinder function (A.10), we obtain

\[ |X_s|1(\tau_t > s) = m(s, W_s; p, t)1(\tau_t > s) \]  
\[ = 1(\tau_t > s) \frac{e^{-\frac{(W_s-y)^2}{4(t-s)}}}{(t-s)^{(p+1)/2}} \]  
\[ \sim 1(\tau_t > s) \frac{e^{-\frac{(W_s-y)^2}{2(t-s)}}}{(t-s)^{(p+1)/2}} \]  
\[ \leq \frac{e^{-\frac{(b(s)-y)^2}{2(t-s)}}}{(t-s)^{(2p+1)/2}} \]  

In particular \( m(t, b(t); p, t) = \lim_{s \uparrow t} m(s, b(t); p, t) = 0 \), for all \( p \) since \( b(t) > y \). Furthermore

\[ \mathbb{E}(1(\tau_t > s)|W_s - y|^p) \leq \int_{b(s)-y}^{\infty} x^p e^{-\frac{(x-y)^2}{2s}} \frac{dx}{\sqrt{2\pi s}} \]  
\[ \leq \frac{1}{\sqrt{2\pi s}} \int_{b(s)-y}^{\infty} x^p e^{-\frac{(x-y)^2}{2t}} dx \]

the last term being finite for all \( s \leq t \). Therefore

\[ \lim_{s \uparrow t} \mathbb{E}|X_s|1(\tau_t > s) \leq C \lim_{s \uparrow t} \frac{e^{-\frac{(b(s)-y)^2}{2(t-s)}}}{(t-s)^{(2p+1)/2}} = 0 \]

since \( b(s)-y \to b(t)-y > 0 \). Furthermore, whenever \( \tau > t \) then \(|X_{\tau}| = |X_t| = \lim_{s \uparrow t} |X_s| = 0 \) since \( \tau > t \) implies \( W_s - y > b(s) - y \) for all \( s < t \) and \( b(t) - y > 0 \). Therefore

\[ \mathbb{E}(|X_{\tau}|) = \mathbb{E}(|X_{\tau}|1(0 < \tau \leq t)) = \mathbb{E}(|m(\tau, b(\tau); p)|1(\tau \leq t)) = \int_0^t |m(s, b(s); p)|F(ds) \]

We already have that \( m(t, b(t); p) = 0 \) for all \( p \). Since \( m \) is a continuous function (because \( b \) is continuous) it follows that the last integral is finite provided that

\[ \int_0^\epsilon |m(s, b(s); p)|F(ds) < \infty \]
for some small positive $\epsilon$. This is the case when $b(0) > -\infty$ since $m(0, b(0); p) < \infty$, so let us assume that $b(0) = -\infty$. Choosing $\epsilon$ small enough and using the asymptotic behavior of the parabolic cylinder function we have:

$$\int_0^\epsilon |m(s, b(s); p)| F(ds) \sim \int_0^\epsilon |(b(s) - y)^{-p-1}| F(ds)$$

(2.12)

and for $p > -1$ the last integral is finite since $(b(s) - y)^{-p-1} \rightarrow 0$ as $s \downarrow 0$ and equals $F(\epsilon)$ for $p = -1$. The case $p < -1$ follows from Lemma 7 and (A.2). Therefore for all $p \in \mathbb{R}$ and $y < b(t)$, we have $\mathbb{E}|X_n| < \infty$ and by Proposition 1 we obtain the class of equations (2.7).

There are several cases of interest, and for some choices of $p$ and $m$, the limit $y \uparrow b(t)$ can be shown to hold. The next step is to investigate what conditions on the boundary $b$ are necessary to allow the limit $y \uparrow b(t)$ in (2.7) to be taken. This limit is not straightforward to compute for all values of the parameter $p$. To see this let us compute the limit as $y \uparrow b(t)$ in equation (2.7) with $p = n = 1$ assuming $b(t)$ is continuously differentiable on $(0, \infty)$ and $b(0) < 0$.

### 2.1.1 Passage to the limit

The next step is to investigate what conditions on the boundary $b$ are necessary to allow the limit $y \uparrow b(t)$ in (2.7) to be taken. This limit is not straightforward to compute for all values of the parameter $p$. To see this let us compute the limit as $y \uparrow b(t)$ in equation (2.7) with $p = n = 1$ assuming $b(t)$ is continuously differentiable on $(0, \infty)$ and $b(0) < 0$. 

The set of integral equations (2.7) reduces to a class of well known equations when $p = n$, a non-negative integer, in which case (2.7) becomes

$$e^{-\frac{y^2}{2t}} H_n(-y/\sqrt{2t}) = \int_0^t e^{-\frac{(b(s)-y)^2}{2(t-s)}} \frac{H_n((b(s) - y)/\sqrt{2(t-s)})}{(t-s)^{n+1/2}} F(ds)$$

(2.13)

where $H_n$ is the Hermite polynomial of degree $n$ (see (A.5)). These equations were given in (1.17) and were derived in e.g. Peskir (2002b) among others. In the next section we examine the limit $y \uparrow b(t)$ which allows the density and boundary to be tightly bound via the integral equations without the appearance of the arbitrary parameter $y$. Afterward, we provide a richer class of examples.
First, in this case, for \( t_0 > 0 \) there exists some \( \epsilon > 0 \) such that \( \epsilon \leq e^{-\frac{(b(s)-b(t))^2}{2(t-s)}} \) for all \( t_0 \leq s \leq t \) since \( \lim_{s \to t} \frac{b(s)-b(t)}{\sqrt{t-s}} = b'(t) = 0 \). Then we have

\[
\epsilon \int_{t_0}^{t} \frac{F(ds)}{\sqrt{t-s}} \leq \int_{t_0}^{t} e^{-\frac{(b(s)-b(t))^2}{2(t-s)}} F(ds) \leq \int_{0}^{t} e^{-\frac{(b(s)-b(t))^2}{2(t-s)}} F(ds)
\]

where the last equality follows from (2.13) with \( n = 0 \). Thus, when \( b(t) \) is differentiable \( \int_{0}^{t} \frac{F(ds)}{\sqrt{t-s}} < \infty \) and therefore \( \int_{0}^{t} \frac{b(t)-b(s)}{(t-s)^{3/2}} F(ds) < \infty \) since \( \frac{b(t)-b(s)}{t-s} \) is finite for all \( 0 \leq s \leq t \) and in the neighborhood of 0 the finiteness follows from Lemma 7 and (A.2) in the Appendix.

Second, for such boundaries, the corresponding density function of \( \tau \) is continuous i.e. \( F(ds) = f(s)ds \) where \( f \) is continuous on \([0, \infty)\) and \( f(0) = 0 \) (see Peskir (2002b) and Peskir (2002a)). As a result,

\[
\frac{e^{-\frac{\nu^2(t)}{t^{3/2}}} b(t)}{t^{3/2}} = \lim_{y \uparrow b(t)} \int_{0}^{t} e^{-\frac{(b(s)-y)^2}{2(t-s)}} \frac{(b(s)-b(t))}{(t-s)^{3/2}} F(ds) + \lim_{y \downarrow b(t)} \int_{0}^{t} e^{-\frac{(b(s)-y)^2}{2(t-s)}} \frac{(b(s)-b(t))}{(t-s)^{3/2}} F(ds) + \lim_{y \uparrow b(t)} 2 \int_{0}^{\infty} 1(u \geq z) \exp \left(-\frac{u^2(1+\frac{b(t)-b(s)}{z\sqrt{t}})^2}{2} \right) f(t-\frac{b(t)-b(s)}{z\sqrt{t}}) du
\]

where we have used the substitutions \( u = \frac{b(t)-y}{\sqrt{t-s}} \) and \( z = \frac{b(t)-y}{\sqrt{t}} \) in the third equality above.

For large \( u \gg z \) we know \( \frac{b(t)-b(s)}{z\sqrt{t}} \approx 0 \) and thus there exists a positive constant \( a < 1 \) such that \( \exp \left(-\frac{u^2(1+\frac{b(t)-b(s)}{z\sqrt{t}})^2}{2} \right) \leq e^{-au^2/2} \) for \( u \gg z \). Therefore, since \( f \) is uniformly
bounded, by the dominated convergence theorem we obtain

$$e^{-\frac{b^2(t)}{2t}b(t)} = \int_0^t e^{-\frac{(b(s)-b(t))^2}{2(t-s)}} \frac{b(s)-b(t)}{(t-s)^{3/2}} f(s) ds$$

$$+ 2 \int_0^\infty \lim_{z \downarrow 0} 1(u \geq z) \exp \left(-\frac{u^2(1 + \frac{b(t-2z^2/u^2)-b(t)^2}{z^2})^2}{2} \right) f(t - t^2/u^2) du$$

$$= \int_0^t e^{-\frac{(b(s)-b(t))^2}{2(t-s)}} \frac{b(s)-b(t)}{(t-s)^{3/2}} f(s) ds + \sqrt{2\pi} f(t)$$

since $$\lim_{z \downarrow 0} \frac{b(t-2z^2/u^2)-b(t)}{z^2} = 0$$. This last equality can be written as:

$$\frac{\phi(b(t)/\sqrt{t})b(t)}{t^{3/2}} = f(t) + \int_0^t \phi \left( \frac{b(t) - b(s)}{\sqrt{t-s}} \right) \frac{(b(s) - b(t))}{(t-s)^{3/2}} f(s) ds$$

(2.14)

This is equation (1.16) derived in Ricciardi et al. (1984) and Peskir (2002b) among others. It demonstrates the complexity involved in exchanging the limit (as $$y \uparrow b(t)$$) and the integral in our new class of integral equations (2.7)– even for the “simple” case $$p = 1$$. Nonetheless, we are able to compute the limiting case for a subclass of integral equations and the next result provides the required conditions on the boundary.

**Theorem 4** Let $$(W_t)_{t \geq 0}$$ be a standard Brownian motion with boundary function $$b \in \mathcal{D}$$. Let $$\tau$$ be the first-passage time of $$W$$ below $$b$$, and let $$F$$ denote its distribution function. Then, for all $$t > 0$$, the following system of integral equations is satisfied:

$$e^{-\frac{|b(t)|^2}{4t}} D_p(-b(t)/\sqrt{t}) = \int_0^t e^{-\frac{(b(s) - b(t))^2}{4(t-s)}} \frac{D_p((b(s) - b(t))/\sqrt{t-s})}{(t-s)^{(p+1)/2}} F(ds)$$

(2.15)

i) For all $$p \leq -1$$

ii) For all $$-1 < p \leq 0$$ whenever $$b$$ is differentiable on $$(0, \infty)$$

iii) For all $$0 < p < 1$$ whenever $$b$$ is continuously differentiable on $$(0, \infty)$$
**Proof.** Note that $D_p(x) > 0$ for all $p \leq 0$. Define

$$k(t) = \lim_{s \uparrow t} \frac{b(s) - b(t)}{\sqrt{t - s}}$$

and

$$g(s; t, y) = e^{-\frac{(b(s) - y)^2}{4(t-s)}} D_p((b(s) - y) / \sqrt{t - s})$$

The function $g$ is a continuous function in $s$ on $0 < s < t$ for all $t > 0$ and $y \leq b(t)$. Thus in order to apply the dominated convergence theorem we will show that $g$ is dominated by an integrable function near $s = 0$ and that $g$ is finite at $s = t$ for all $y \leq b(t)$. First note that when $b(0)$ is finite then $|g(0; t, y)|$ exists for all $p$ and $y \leq b(t)$ and when $b(0) = -\infty$ then

$$\int_0^{\epsilon(p)} \frac{|g(s; t, y)|}{(t-s)^{(p+1)/2}} F(ds) \sim \int_0^{\epsilon(p)} (b(s) - y)^{-p-1} F(ds) < \infty$$

for some $\epsilon(p) > 0$ and all $p$, $y \leq b(t)$. The finiteness of the last integral follows from the fact that the integrand $(b(s) - y)^{-p-1}$ is a monotone continuous function in $y$ and thus for some $y_*$ near $b(t)$ it is dominated by $(b(s) - y_*)^{-p-1}$ which is integrable on $(0, \epsilon(p)]$ by Lemma 7. Thus we only need to show $\lim_{s \uparrow t} g(s; t, b(t)) (t-s)^{(p+1)/2} < \infty$ in order to apply the dominated convergence theorem since $\lim_{s \uparrow t} g(s; t, y) (t-s)^{(p+1)/2} = 0$ for $y < b(t)$.

i) Since $\lim_{s \uparrow t} (t-s)^{-(p+1)/2} = 0$ the case $|k(t)| < \infty$ is straightforward. Suppose $k(t) = \infty$.

Then for $s$ close to $t$,

$$g(s; t, b(t)) \sim e^{-\frac{(b(s) - b(t))^2}{2(t-s)}} \left(\frac{b(s) - b(t)}{\sqrt{t-s}}\right)^p \rightarrow 0$$

using the asymptotic behavior of $D_p(x)$ for large $x$. Similarly, suppose $k(t) = -\infty$ then the asymptotic behavior of $g(s; t, b(t)) (t-s)^{-(p+1)/2}$ is

$$g(s; t, b(t)) (t-s)^{-(p+1)/2} \sim \left(-\frac{b(s) - b(t)}{\sqrt{t-s}}\right)^{-1-p} (t-s)^{-(p+1)/2} = (b(t) - b(s))^{-1-p} \downarrow 0$$
as \( s \uparrow t \) since \(-p - 1 \geq 0\) and \( b \) is continuous. Therefore, taking the limit \( y \uparrow b(t) \) in (2.7), by the dominated convergence theorem the result follows.

ii) We showed that when \( b \) is differentiable (and thus continuous) then \( \int_{t_0}^{t} \frac{F(ds)}{\sqrt{t-s}} < \infty, \ t_0 > 0 \). Furthermore, differentiability implies \( k(t) = 0 \). Similarly as in part i) we see that \( \lim_{s \uparrow t} g(s; t, y)/(t-s)^{p/2} = 0 \) for all \( y \leq b(t) \) and thus \( g(s; t, y)/(t-s)^{p/2} \) is bounded on \([t_0, t]\). By the dominated convergence theorem we can exchange the limit and the integral in 2.15.

iii) When \( b \) is continuously differentiable on \((0, \infty)\) then \( f \) is continuous on \((0, t]\) for all \( t > 0 \) Peskir (2002b) and so \( \int_0^t \frac{F(ds)}{(t-s)^{(p+1)/2}} ds < \infty \) since \( 0 < (p + 1)/2 < 1 \). Furthermore, \( |g(s; t, y)| \) is bounded for \( 0 < s \leq t \) for all \( y \leq b(t) \) since \( k(t) = 0 \). The result follows by the dominated convergence theorem. \( \square \)

Note that the differentiability condition on the boundary in part ii) can be relaxed to \(|k(t)| < \infty\) for all \( t > 0 \). In this case we still have \( \int_{t_0}^{t} \frac{F(ds)}{\sqrt{t-s}} < \infty, \ t_0 > 0 \), using the same argument as before and the proof of part ii) is still valid. Also, it would be straightforward to extend the class of equations (2.7) and (2.15) to the class of equations with a complex valued parameter \( p \).

We end this subsection with a simple argument which shows an alternative and straightforward derivation of subclasses of the two sets of integral equations (2.7) and (2.15). Let us denote the class of equations (2.7) by \( \{A_p(y, t)\}_{p \in \mathbb{R}}, \ y < b(t) \), and the class (2.15) by \( \{B_p(t)\}_{p<1} \). Write \( y = z - \theta, \ z < b(t), \ \theta > 0 \), and let \( Q(\theta) \) be a \( \sigma \)-finite positive measure on \([0, \infty)\). We saw above that when \( Q(d\theta) = \theta^{-p-1} d\theta, \ p < 0 \) we can obtain equations \( \{A_p(z, t)\}_{p<0} \) from \( A_0(z - \theta, t) \) by applying the integral transform (2.2) defined in Theorem 2 w.r.t. the measure \( Q \). Furthermore, in the same way we can obtain equations \( \{B_p(t)\}_{p \leq -1} \) from equation \( A_0(b(t) - \theta, t) \) since \( Q(0) < \infty \) for \( p \leq -1 \). In both cases it is sufficient to assume that the boundary \( b \) is continuous for equation \( A_0(z - \theta, t), \ z \leq b(t), \ \theta > 0 \) to hold for all \( t > 0 \). Thus, the first part of the above corollary can be obtained by a simple integra-
tion without passing to the limit \( y \uparrow b(t) \). Moreover, by the same integration technique, we can obtain equation \( \{ B_p(t) \}_{-1 < p < 0} \) from \( B_0(t) \) under the hypothesis of Theorem 4. However, the martingale technique employed in the derivation of these equations is more general and does not require an integral representation of the kernel function in the resulting integral equation.

### 2.1.2 Special Cases

For different values of \( p \) the parabolic cylinder function, \( D_p \), can be written in terms of other special functions. When \( p \) is a non-negative integer we already saw the connection with the Hermite polynomials which can be written in terms of the Laguerre polynomials. The case when \( p \) is a negative integer covers the system of equations (1.18) derived in Peskir (2002b) as we will see below. Therefore the classes \( \{ A_p(y, t) \}_{p \in \mathbb{R}} \) and \( \{ B_p(t) \}_{p < 1} \) contain all currently known Volterra equations of the first kind which arise from the FPT for Brownian motion. Furthermore, these equations can be written in terms of the Whittaker function (see (A.6)) or confluent hypergeometric functions using the representation of the parabolic cylinder function for all values of \( p \). For \( p = -1/2 \) there is also a connection with the modified Bessel function of the third kind \( K_\nu \) (see Case 4 below). Cases 1, 2 and 3 below discuss the currently known Volterra equations while cases 4 and 5 are examples of new Volterra equations.

**Case 1: \( p = 0 \)**

In this case (2.15) becomes

\[
\int_0^t \frac{e^{-\frac{(b(t)-b(s))^2}{2(t-s)}}}{\sqrt{t-s}} F(ds) = \frac{e^{-b(t)^2/2t}}{\sqrt{t}}
\]
which can be written as
\[
\int_0^t \frac{1}{\sqrt{t-s}} \phi \left( b(t) - b(s) \right) F(ds) = \frac{1}{\sqrt{t}} \phi(b(t)/\sqrt{t})
\] (2.16)

This equation was already discussed in Chapter 1.

**Case 2:** \( p = -1 \)

In this case (2.15) becomes
\[
\int_0^t \Phi \left( b(t) - b(s) \right) F(ds) = \Phi(b(t)/\sqrt{t})
\] (2.17)

This is equation (1.9) of Chapter 1.

**Case 3:** \( p = -n, \ n = 1, 2, 3... \)

In this case (2.15) becomes
\[
\int_0^t e^{-(b(t) - b(s))^2/4} D_n \left( \frac{b(s) - b(t)}{\sqrt{t-s}} \right) (t-s)^{n-1/2} F(ds) = \frac{e^{-b(t)^2/4}}{\sqrt{2\pi}} D_n \left( -\frac{b(t)}{\sqrt{t}} \right) t^{n-1/2}
\] (2.18)

We claim that (2.18) is equivalent to the system of equations derived in Peskir (2002b) and given in (1.18). To show this, consider the kernel of the integral equation (2.18), which is of the form \( \frac{1}{\sqrt{2\pi}} e^{-x^2/4} D_n(-x) =: G_n(x) \). Using (A.13) we have
\[
\frac{d}{dx} G_{n+1}(x) = G_n(x)
\]
and thus
\[
G_{n+1}(x) = \int_{-\infty}^x G_n(u) du + C
\]
Taking \( x = 0 \) and using (A.14) we see that \( C = 0 \). Therefore we can rewrite (2.18) as
\[
\int_0^t G_n \left( \frac{b(t) - b(s)}{\sqrt{t-s}} \right) (t-s)^{(n-1)/2} F(ds) = G_n(b(t)/\sqrt{t}) t^{(n-1)/2}
\]
where \( n = 0, 1, 2, \ldots \) and \( G_n \) satisfies the recursion formula
\[
G_{n+1}(x) = \int_{-\infty}^{x} G_n(u) du
\]
with \( G_0(x) = \phi(x) \) since \( D_0(-x) = e^{-x^2/4} \). Therefore, the system of integral equations (2.18) is equivalent to the system of equations (1.18). This completes the proof of the above claim.

The next two cases provide two new integral equations arising as specific cases of our general class:

**Case 4**: \( p = -1/2 \). In this case, using (A.7), (2.15) becomes
\[
\sqrt{\frac{b(t)}{t}} e^{-\frac{b^2(t)}{4t}} K_{1/4} \left( \frac{b^2(t)}{4t} \right) = \int_0^t \sqrt{\frac{b(t) - b(s)}{t - s}} e^{-\frac{(b(t) - b(s))^2}{4(t-s)}} K_{1/4} \left( \frac{(b(t) - b(s))^2}{4(t-s)} \right) F(ds) \tag{2.19}
\]

**Case 5**: For the case \( p = -2 \) and using the results for \( p = 0 \) and \( p = -1 \) together with (A.8), (2.15) becomes
\[
\int_0^t \left[ \frac{s}{\sqrt{t-s}} \phi \left( \frac{b(t) - b(s)}{\sqrt{t-s}} \right) + b(s) \Phi \left( \frac{b(t) - b(s)}{\sqrt{t-s}} \right) \right] F(ds) = 0 \tag{2.20}
\]

The last equation is a special case of a new class of equations that can be derived from (2.15) using the recursive relation property (A.12) of the parabolic cylinder function. Using
this relation, the class (2.15) for \( p \leq -1 \), can be written as:

\[
\begin{align*}
e^{-\frac{b(t)^2}{4t}} \bigg\{ & D_{p+1}(-b(t)/\sqrt{t}) + \frac{b(t)}{\sqrt{t}} D_p(-b(t)/\sqrt{t}) \bigg\} \\
= & \int_0^t e^{-\frac{(b(s)-b(t))^2}{4(t-s)}} \left\{ D_{p+1} \left( \frac{b(s)-b(t)}{\sqrt{t-s}} \right) - \frac{b(s)-b(t)}{\sqrt{t-s}} D_p \left( \frac{b(s)-b(t)}{\sqrt{t-s}} \right) \right\} F(ds) \\
= & \int_0^t e^{-\frac{(b(s)-b(t))^2}{4(t-s)}} \frac{(t-s)(p+2)D_{p+1} \left( \frac{b(s)-b(t)}{\sqrt{t-s}} \right)}{(t-s)(p+1)D_p \left( \frac{b(s)-b(t)}{\sqrt{t-s}} \right)} F(ds) \\
= & e^{-\frac{b(t)^2}{4t}} \bigg\{ D_{p+1}(-b(t)/\sqrt{t}) + \frac{b(t)}{\sqrt{t}} D_p(-b(t)/\sqrt{t}) \bigg\} \\
= & \int_0^t s \left( \frac{b(s)-b(t)}{\sqrt{t-s}} \right) D_{p+1} \left( \frac{b(s)-b(t)}{\sqrt{t-s}} \right) F(ds) \\
= & \int_0^t e^{-\frac{(b(s)-b(t))^2}{4(t-s)}} \frac{b(s)D_p \left( \frac{b(s)-b(t)}{\sqrt{t-s}} \right)}{(t-s)(p+1)D_p \left( \frac{b(s)-b(t)}{\sqrt{t-s}} \right)} F(ds)
\end{align*}
\]

Thus, from the last equality, we obtain the class of equations:

\[
\begin{align*}
\int_0^t e^{-\frac{(b(s)-b(t))^2}{4(t-s)}} \left\{ \frac{s}{\sqrt{t-s}} D_{p+1} \left( \frac{b(s)-b(t)}{\sqrt{t-s}} \right) + b(s)D_p \left( \frac{b(s)-b(t)}{\sqrt{t-s}} \right) \right\} F(ds) = 0 \quad (2.21)
\end{align*}
\]

The above system of equations holds whenever the system (2.15) holds. A similar class of equations can constructed for all values of \( p \) from the class \( \{A_p(y, t)\}_{p \in \mathbb{R}} \).

### 2.1.3 Uniqueness of a solution

In this section we examine sufficient conditions for the boundary \( b \) such that the class of integral equations \( \{B_p\}_{p<1} \) has a unique continuous solution. In order to guarantee the existence of a continuous density function \( f \) which solves the system \( \{B_p\}_{p<1} \) we need the following assumptions on the boundary. Suppose that \( b(.) \) is continuously differentiable on
(0, T] and assume \( \lim_{t \to 0} |b'(t)| t^\epsilon < \infty \) for some \( 0 < \epsilon < 1/2 \). Then, \( \lim_{t \to 0} |b'(t)| t^\epsilon < \infty \) implies \(-\infty < b(0) < 0 \) (since \( b \) is a regular boundary and \( \epsilon < 1/2 \)) and therefore \( f(0) = 0 \) (Peskir (2002a)). Together with the continuous differentiability of the boundary we have that \( F(ds) = f(s)ds \) where \( f \) is continuous on \([0, \infty)\). Denote \((p+1)/2 = \lambda\) so that \( \lambda < 1 \) and let

\[
g_{2\lambda+1}(t) = e^{-\frac{b(t)^2}{4t}} D_{2\lambda+1}(-b(t)/\sqrt{t})/t^\lambda
\]

\[
K_{2\lambda+1}(t, s) = e^{-\frac{(b(s)-b(t))^2}{4(t-s)}} D_{2\lambda+1}((b(s) - b(t))/\sqrt{t-s})
\]

Then the class \( \{B_{2\lambda+1}\}_{\lambda < 1} \) of integral equations can be written as:

\[
g_{2\lambda+1}(t) = \int_0^t \frac{K_{2\lambda+1}(t, s)}{(t-s)^\lambda} f(s) ds
\]

We know that the above equation has a continuous solution given by the first passage time density function \( f \) (Peskir (2002b)) under the current assumptions on the boundary. The following result shows that these assumptions are also sufficient for \( f \) to be the unique solution.

**Theorem 5** For each \( T > 0 \) let \( b(t) \) be a regular boundary, continuously differentiable on \((0, T]\), and satisfying \( |b'(t)| = O(t^{-\epsilon}) \) for some \( 0 < \epsilon < 1/2 \) and all sufficiently small \( t \). Then \( \tau \) has a density function, \( f \), given as the unique continuous on \([0, T]\) solution of the integral equation

\[
e^{-\frac{b(t)^2}{4t}} D_p(-b(t)/\sqrt{t}) = \int_0^t e^{-\frac{(b(s)-b(t))^2}{4(t-s)}} D_p((b(s) - b(t))/\sqrt{t-s})/ (t-s)^{(p+1)/2} f(s) ds
\]

for any \(-1 < p < 1\).

**Proof.** Recognizing equation (2.22) as a generalized Abel equation of the first kind (when \( 0 < \lambda < 1 \)), we can employ the standard results on integral equations of this class (see Bocher
(1909)) to show that \( f \) is the unique solution. To show uniqueness we transform equation 2.22 to a new Volterra equation of the first kind which after differentiation w.r.t. \( t \) reduces to a Volterra equation of the second kind. Then, the latter equation is shown to have a unique continuous solution.

Using Lemma (8) we have, \( |D_p(\frac{b(t)-b(s)}{\sqrt{t-s}})| < M_p \) for some \( M_p > 0 \) and all \( 0 \leq s \leq t, \ p \in \mathbb{R} \). Applying Abel’s transform to equation (2.22) we obtain:

\[
\int_0^u \frac{g_{2\lambda+1}(t)}{(u-t)^{1-\lambda}} dt = \int_0^u \int_s^u \frac{K_{2\lambda+1}(t,s)}{(u-t)^{1-\lambda}(t-s)^{\lambda}} df(s) ds
\]

(2.23)

where we have used Fubini’s theorem (since \( |K_{2\lambda+1}| \) is bounded) to exchange the order of integration. Let

\[
\bar{g}_\lambda(u) := \int_0^u \frac{g_{2\lambda+1}(t)}{(u-t)^{1-\lambda}} dt
\]

\[
\bar{K}_{2\lambda+1}(u,s) := \int_s^u \frac{K_{2\lambda+1}(t,s)}{(u-t)^{1-\lambda}(t-s)^{\lambda}} dt = \int_0^1 \frac{K_{2\lambda+1}(y(u-s) + s,s)}{(1-y)^{1-\lambda}y^\lambda} dy
\]

then equation (2.23) can be written as:

\[
\bar{g}_\lambda(u) = \int_0^u \bar{K}_{2\lambda+1}(u,s)f(s) ds
\]

(2.24)

Next, we apply the standard technique of differentiation on \( u \) to reduce (2.24) to a Volterra equation of the second kind. First we show that \( g_{2\lambda+1}(0) = 0 \) and that \( \bar{g}_\lambda(u) \) has a continuous derivative for all \( u \geq 0 \). For any \( \lambda_1, \lambda_2 \in R \), since \( b(t)/\sqrt{t} \downarrow -\infty \) when \( t \downarrow 0 \) for a regular boundary \( b \), we have (using the asymptotic expansion of the parabolic cylinder function)

\[
\lim_{t \downarrow 0} \frac{g_{2\lambda_1+1}(t)}{t^{\lambda_2}} = \lim_{t \downarrow 0} e^{-b^2(t)/(2t)} \left( -\frac{b(t)}{\sqrt{t}} \right)^{2\lambda_1+1} = \lim_{t \downarrow 0} e^{-b^2(t)/2t} (-b(t))^{2\lambda_1+1} \left( \frac{1}{t} \right)^{2\lambda_1+\lambda_2+1/2} = 0
\]
since $\infty > -b(0) > 0$. In particular $g_{2\lambda+1}(0) = 0$. Also, using (A.13),

$$
dg_{2\lambda+1}(t)/dt = -\frac{\lambda g_{2\lambda+1}(t)}{t} + \sqrt{t}g_{2\lambda+2}(t) \left( \frac{b'(t)}{\sqrt{t}} - \frac{b(t)}{2t^{3/2}} \right)$$

(2.25)

$$
dg_{2\lambda+1}(t)/dt = -\frac{\lambda g_{2\lambda+1}(t)}{t} + \frac{g_{2\lambda+2}(t)}{t^\epsilon}b'(t)t^\epsilon - \frac{b(t)g_{2\lambda+2}(t)}{2t}
$$

(2.26)

Under our assumption on the boundary and since $b(0) > -\infty$ each term in the last line goes to 0 as $t \downarrow 0$ and we obtain

$$
\lim_{t \downarrow 0} g'_{2\lambda+1}(t) = 0.
$$

Therefore, since $b$ is continuously differentiable, it follows that $g'_{2\lambda+1}(t)$ and $g_{2\lambda+1}(t)$ are continuous functions for all $t \geq 0$ and since $g_{2\lambda+1}(0) = 0$, by Bocher (1909) (Theorem 3, p.5), $\tilde{g}_\lambda(u)$ has a continuous derivative, for all $u \geq 0$, given by

$$
\tilde{g}_\lambda'(u) = \int_0^u \frac{g_{2\lambda+1}(t)}{(u-t)^{1-\lambda}} dt
$$

Next we compute the derivative, w.r.t. $u$, of the right hand side of (2.24). Since

$$
|D_{2\lambda+1}((b(s) - b(t))/\sqrt{t-s})| < C_\lambda
$$

for some $C_\lambda > 0$ and all $0 \leq s \leq t$, it follows that $K_{2\lambda+1}(y(u-s)+s, s) < C_\lambda$ for all $0 \leq s \leq u$ and $0 \leq y \leq 1$, while

$$
\int_0^1 C_\lambda \frac{1}{(1-y)^{1-\lambda}y^{\lambda}} dy = C_\lambda B(1-\lambda, \lambda)
$$

where $B(\cdot, \cdot)$ is the Beta function. Thus, by the dominated convergence theorem,

$$
\tilde{K}_{2\lambda+1}(u, u) = \lim_{s \uparrow u} \tilde{K}_{2\lambda+1}(u, s)
$$

$$
= \int_0^1 \lim_{s \uparrow u} K_{2\lambda+1}(y(u-s)+s, s) \frac{1}{(1-y)^{1-\lambda}y^{\lambda}} dy
$$

$$
= D_{2\lambda+1}(0)B(1-\lambda, \lambda) \neq 0
$$
for all \( u \geq 0 \). Furthermore, using Lemma (8),

\[
y^{1/2-\epsilon}|dK_{2\lambda+1}(y(u - s) + s, s)/du| = e^{-\frac{(b(u)-b(y(u-s)+s))^2}{4y(u-s)}} D_{2\lambda+2}(b(u) - b(y(u-s)+s))/y(u-s)}^{\epsilon+1/2} ((y(u - s))\frac{y(u-s)}{(y(u-s)+s)}^\epsilon) \times \]

\[
\times (b'(y(u-s) + s)(y(u-s) + s)^\epsilon - \frac{(b(y(u-s) + s) - b(s))(y(u-s) + s)^\epsilon)}{2y(u-s)}
\]

\[
\frac{|H_\epsilon(y, u, s)|}{(u - s)\epsilon+1/2} \leq \frac{M}{(u - s)\epsilon+1/2}
\]

for some constant \( M > 0 \) and for all \( 0 \leq s \leq u \) and \( 0 \leq y \leq 1 \). Also

\[
\frac{\int_0^u \left( \int_0^1 \frac{M}{(u - s)^\epsilon+1/2} \frac{1}{(1-y)^{1-\lambda}y^{-1/2-\epsilon}} \right) f(s) ds}{\int_0^u \frac{f(s)}{(u - s)^{1/2+\epsilon}} ds} < \infty
\]

since \( f \) is continuous on \([0, T]\). Thus the derivative \( dK_{2\lambda+1}(y(u - s) + s, s)/du \) is dominated by an integrable function. Denote

\[
K_{2\lambda+1}^\delta(y, u, s) := K_{2\lambda+1}(y(u - s) + s + \delta y, s) - K_{2\lambda+1}(y(u - s) + s, s)
\]

for \( \delta > 0 \) and note that \( K_{2\lambda+1}^\delta(y, u, s) / \delta \to dK_{2\lambda+1}(y(u - s) + s, s)/du \) uniformly on \((s, y) \in [0, u] \times [0, 1]\) as \( \delta \downarrow 0 \). Therefore, if \( \mu \) denotes the measure on \([0, u]\) with Radon-Nykodim derivative \( f \) and \( \nu \) denotes the measure on \([0, 1]\) with Radon-Nykodim derivative \( \frac{1}{(1-y)^{1-\lambda}y^{-1/2-\epsilon}} \), by Fubini’s theorem (applied twice) and the dominated convergence theo-
rems, we have

\[
\lim_{\delta \downarrow 0} \int_0^u \int_0^1 y^{1/2-\epsilon} \frac{K_{2\lambda+1}^\delta(y, u, s)}{\delta} \frac{f(s)}{(1 - y)^{1-\lambda}y^{-(1/2-\epsilon)}} dy ds
\]

\[
= \lim_{\delta \downarrow 0} \int_{[0,u] \times [0,1]} y^{1/2-\epsilon} \frac{K_{2\lambda+1}^\delta(y, u, s)}{\delta} d(\mu \times \nu)
\]

\[
= \int_{[0,u] \times [0,1]} \lim_{\delta \downarrow 0} y^{1/2-\epsilon} \frac{K_{2\lambda+1}^\delta(y, u, s)}{\delta} d(\mu \times \nu)
\]

\[
= \int_{[0,u] \times [0,1]} \frac{H_\lambda^\epsilon(y, u, s)}{(u - s)^{\epsilon+1/2}} d(\mu \times \nu)
\]

\[
= \int_0^u \left( \int_0^1 \frac{H_\lambda^\epsilon(y, u, s)}{(1 - y)^{1-\lambda}y^{-(1/2-\epsilon)}} dy \right) \frac{f(s)}{(u - s)^{\epsilon+1/2}} ds
\]

where the quantity

\[
R_\lambda^\epsilon(u, s) := \int_0^1 \frac{H_\lambda^\epsilon(y, u, s)}{(1 - y)^{1-\lambda}y^{-(1/2-\epsilon)}} dy
\]

is bounded by \(MB(1 - \lambda, \lambda - (1/2 - \epsilon))\) and continuous for all \(0 \leq s < u\) with a possible discontinuity at \(u = s\). Therefore the derivative of (2.24) w.r.t. \(u\) is given by

\[
\bar{g}_\lambda'(u) = \lim_{\delta \downarrow 0} \left( \int_0^{u+\delta} \tilde{K}_{2\lambda+1}(u + \delta, s) f(s) ds - \int_0^{u} \tilde{K}_{2\lambda+1}(u, s) f(s) ds \right)
\]

\[
= \lim_{\delta \downarrow 0} \frac{1}{\delta} \int_u^{u+\delta} \tilde{K}_{2\lambda+1}(u + \delta, s) f(s) ds
\]

\[
+ \lim_{\delta \downarrow 0} \int_0^{u} \frac{\tilde{K}_{2\lambda+1}(u + \delta, s) - \tilde{K}_{2\lambda+1}(u, s)}{\delta} f(s) ds
\]

\[
= \tilde{K}_{2\lambda+1}(u, u) f(u) + \int_0^{u} \frac{R_\lambda^\epsilon(u, s)}{(u - s)^{\epsilon+1/2}} f(s) ds
\]

Thus we obtained the Volterra equation of the second kind:

\[
\frac{\bar{g}_\lambda'(u)}{\tilde{K}_{2\lambda+1}(u, u)} = f(u) + \int_0^{u} \frac{R_\lambda^\epsilon(u, s)}{\tilde{K}_{2\lambda+1}(u, u)(u - s)^{\epsilon+1/2}} f(s) ds \quad (2.27)
\]

Since \(R_\lambda^\epsilon\) is finite on \(0 \leq s \leq u\) with a possible discontinuity only along the curve \(s = u\) and since \(f\) is continuous on \([0, \infty)\), by Bocher (1909) (Theorem 3, p. 19), (2.27) has a unique
continuous solution. Thus, the continuous solution to (2.22) is unique. Therefore, for each 
$-1 < p < 1$ equation $B_p$ has a unique continuous solution. This completes the proof. □

Finally, we show that any solution to the system $\{B_p\}_{p \leq -1}$ is also a solution to $B_0$.

**Corollary 1** For each $T > 0$ let $b(t)$ be a regular boundary, continuously differentiable on $(0,T]$, and satisfying $|b'(t)| = O(t^{-\epsilon})$ for some $0 < \epsilon < 1/2$ and all sufficiently small $t$. Then $\tau$ has a density function, $f$, given as the unique continuous on $[0,T]$ solution of the system of integral equations

$$
\frac{e^{-\frac{b(t)^2}{4t}}D_p(-b(t)/\sqrt{t})}{t^{(p+1)/2}} = \int_0^t e^{-\frac{(b(s)-b(t))^2}{4(t-s)}} \frac{D_p((b(s)-b(t))/\sqrt{t-s})}{(t-s)^{(p+1)/2}} f(s)ds
$$

for all $p \leq -1$.

**Proof.** Suppose $g : [0,T] \rightarrow \mathbb{R}$ is any continuous solution to $B_p$, for all $p \leq -1$, satisfying the hypothesis of Theorem 5. Then, using the integral representation of $D_p$, given in (A.4), and the fact that $D_p(x) > 0$, $\forall p < 0$, by Fubini’s theorem we can write equation $B_p$ as

$$
\int_0^t \int_0^\infty e^{-\frac{(b(x)-b(s))^2}{4(t-s)}} u^{-p-1} \frac{g(s)}{\sqrt{t-s}} ds du = \int_0^\infty u^{-p-1} \frac{e^{-\frac{(u\sqrt{t}-b(t))^2}{2t}}}{\sqrt{t}} du
$$

and

$$
\int_0^\infty u^{-p-1} \int_0^t e^{-\frac{(b(x)-b(s))^2}{4(t-s)}} u^{-p-1} \frac{g(s)}{\sqrt{t-s}} ds du = \int_0^\infty u^{-p-1} \frac{e^{-\frac{(u\sqrt{t}-b(t))^2}{2t}}}{\sqrt{t}} du
$$

The last equality is an equality of Mellin transforms. Thus, from the uniqueness of the Mellin transform we obtain equation:

$$
\int_0^t e^{-\frac{(b(x)-b(s))^2}{4(t-s)}} \frac{g(s)}{\sqrt{t-s}} ds = \frac{e^{-\frac{(u\sqrt{t}-b(t))^2}{2t}}}{\sqrt{t}}
$$

which holds for all $u > 0$. In order to show that the last equation also holds for $u = 0$ we take the limit $u \downarrow 0$. Since the term $e^{-\frac{(b(x)-b(s))^2}{4(t-s)}}$ is bounded we can exchange the
integral and the limit in the above equation to obtain

\[ \int_0^t e^{-\left(\frac{b(s) - b(t)}{\sqrt{t-s}}\right)^2/2} \frac{g(s)}{\sqrt{t-s}} ds = \frac{e^{b(t)^2/(2t)}}{\sqrt{t}}. \]

Since the last equation has a unique continuous solution, as shown above, it follows that \( g(t) = f(t) \) (assuming the boundary satisfies the hypothesis of Theorem 5). Thus any continuous solution to the system \( \{B_p\}_{p \leq -1} \) is also a solution to \( B_0 \) which has a unique continuous solution. \( \square \)

### 2.1.4 Functional Transforms

Next we consider some functional transforms of the boundary and the corresponding density functions. The new density functions can be easily expressed in terms of the original boundary and its density function using equation \( B_0 \) and Theorem 5. Furthermore, applying the functional transforms successively we can obtain a larger class of boundary functions with known probability density function.

Suppose \( b \) satisfies the hypotheses of Theorem 5 and thus has a corresponding continuous density function \( f \) and introduce the functional transforms:

\[ (T_1^\alpha . b)(t) = b(t) + \alpha t, \ \alpha \in \mathbb{R} \quad (2.29) \]
\[ (T_2^\gamma . b)(t) = b(\gamma t)/\sqrt{\gamma}, \ \gamma > 0 \quad (2.30) \]
\[ (T_3^\beta . b)(t) = (1 + \beta t)b \left( \frac{t}{1 + \beta t} \right), \ \beta \geq 0 \quad (2.31) \]

Note that we can set \( \beta < 0 \) with \( t \leq -1/\beta \) in the last transform. Moreover, \( (T_1 . b), (T_2 . b) \) and \( (T_3 . b) \) all satisfy the hypotheses of Theorem 5. Denote with \( f_1, f_2 \) and \( f_3 \), respectively the corresponding density functions of the first-passage times of \( W_t \) to these boundaries.
Combining the three transforms into the single transform

\[(T.b)(t) := \frac{1 + \beta \gamma t}{\sqrt{\gamma}} b \left( \frac{\gamma t}{1 + \beta \gamma t} \right) + \alpha t, \alpha \in \mathbb{R}, \beta \geq 0, \gamma > 0\]

we have the following result for the corresponding density function \(f_T\).

**Lemma 2** For each \(S > 0\) let \(b(t)\) be a regular boundary, continuously differentiable on \((0, S]\), and satisfy \(|b'(t)| = O(t^{-\epsilon})\) for some \(0 < \epsilon < 1/2\) in the neighbourhood of zero. Let \(f\) be the continuous density function of the first-passage time of the standard Brownian motion \(W_t\) to \(b(t)\). Let \(\tilde{b}(t) = (T.b)(t)\). Then the first-passage time to the boundary \(\tilde{b}(t)\) has a continuous density, \(\tilde{f}\), given by:

\[
\tilde{f}(t) = \frac{\gamma f\left( \frac{\gamma t}{1 + \beta \gamma t} \right)}{(1 + \beta \gamma t)^{3/2}} e^{-\left(1 + \beta \gamma t\right)b\left( \frac{\gamma t}{1 + \beta \gamma t} \right)(\beta b\left( \frac{\gamma t}{1 + \beta \gamma t} \right)/2 + \alpha/\sqrt{\gamma} - \alpha^2 t/2}
\]

(2.32)

**Proof.** Since \(b(t)\) is continuously differentiable then so is \(\tilde{b}(t)\) and thus the first-passage time of \(W_t\) to \(\tilde{b}(t)\) has a continuous density function. Using equation \(B_0\) we can easily find the relations between \(f\) and \(f_i, i = 1, 2, 3\).

For \(f_1\) we have:

\[
e^{-\left( b(t) + \alpha t \right)^2/(2t)} \frac{1}{\sqrt{t}} = \int_0^t e^{-\left( b(t) - b(s) + \alpha \left[ (t-s) \right] \right)^2/(2(t-s))} f_1(s) ds
\]

\[
e^{-b(t)^2/(2t)} \frac{1}{\sqrt{t}} = \int_0^t e^{-\left( b(t) - b(s) \right)^2/(2(t-s))} \frac{e^{ab(s) + \alpha^2 s}/2}{\sqrt{t-s}} f_1(s) ds
\]

Therefore, due to the uniqueness of solutions (Theorem 5), we must have

\[f_1(t) = f(t)e^{-ab(t) - \alpha^2 t/2} = e^{-\alpha\left( T_{\alpha}^{\beta/2},b \right)(t) f(t)\}}
\]

(2.33)
For \( f_2 \) we obtain:

\[
e^{-b(\gamma t)^2/(2\gamma t)} \sqrt{t} = \int_0^t e^{-\frac{(b(\gamma t) - b(s))^2}{2(t-s)}} f_2(s)ds
\]

\[
e^{-b(u)^2/(2u)} \sqrt{u} = \int_0^u e^{-\frac{(b(u) - b(x))^2}{2(u-x)}} f_2(x/\gamma)/\gamma dx
\]

and, applying Theorem 5,

\[
f_2(t) = \gamma f(\gamma t) = \gamma^{3/2}(T_{2,\gamma}^\beta f)(t)
\]  \hspace{1cm} (2.34)

For \( f_3 \) we obtain:

\[
e^{-\frac{1}{2}(1+\beta t)^2 b(\frac{t}{1+\beta t})^2} \sqrt{t} = \int_0^t \exp \left( -\frac{1}{2(t-s)} \left( (T_{3}^\beta b)(t) - (T_{3}^\beta b)(s) \right)^2 \right) \frac{f_3(s)}{\sqrt{t-s}} ds
\]

\[
e^{-\frac{\beta^2(u)}{2(1-\beta u)}} \sqrt{u} = \int_0^u \exp \left( -\frac{1-\beta x}{2(u-x)(1-\beta u)} (b(u) - b(x) \frac{1-\beta u}{1-\beta x})^2 \right) \frac{(1-\beta x)^{-3/2} f_3(\frac{x}{1-\beta x})}{\sqrt{u-x}} dx
\]

\[
e^{-\frac{b(u)^2}{2u}} \sqrt{u} = \int_0^u e^{-\frac{(b(u) - b(x))^2}{2(u-x)}} e^{\frac{\beta^2(x)}{2(1-\beta x)} (1-\beta x)^{-3/2} f_3(\frac{x}{1-\beta x})} dx
\]

where we have made the substitution \( u = \frac{t}{1+\beta t}, \ x = \frac{s}{1+\beta s} \). Therefore, by Theorem 5,

\[
f_3(t) = f \left( \frac{t}{1+\beta t} \right) \exp(-\beta(1+\beta t)b^2((\frac{t}{1+\beta t})^2/2)(1+\beta t)^{-3/2})
\]  \hspace{1cm} (2.35)

Applying the transforms \( T_3^\beta, T_2^\gamma, T_1^\alpha \) successively to \( b(t) \) transforms \( f(t) \) to \( \tilde{f}(t) \) and (2.32) follows from Theorem 5. \( \Box \)

The result for \( f_1 \) can alternatively be obtained by a simple measure change argument based on Girsanov’s theorem. The measure change argument is used in the next section to derive
a class of Fredholm equations. The result for $f_2$ can alternatively be derived through the time change $t \rightarrow \alpha t$ applied directly to $\tau$ using the scaling property of the Brownian motion. The time change is explored in more detail in Section 5.3. The result for $f_3$ was originally obtained by Alili and Patie (2005) for more general boundaries, using probabilistic arguments and the properties of the Brownian bridge. Nevertheless, it is instructive and pleasing that the integral equations combined with Theorem 5 lead to a unifying derivation of all of these transformation results.

2.2 Fredholm Equations

Similarly to Section 2.1, in this section we examine the well known martingale $e^{-\alpha W_t - \alpha^2 t/2}$ which gives rise to a Fredholm integral equation of the first kind. This equation is used to obtain alternative derivation of known closed form results for the linear, quadratic and square-root boundaries and is a major building block for the results of Chapter 3. Furthermore, as we will see, this equation is simply the Laplace transform of the integral equations (2.7) for a particular class of boundaries.

We assume that $b$ is continuous on $[0, \infty)$. Let $\tau_\alpha = \inf \{ t > 0; W_t \leq b(t) + \alpha t \}$ with cumulative distribution function $F_\alpha$ and define the set

$$\mathcal{A}_{b(t)} := \{ \alpha \in \mathbb{R}; b_\alpha(t) := b(t) + \alpha t \geq c \text{ for some } c < 0 \text{ and all } t \geq 0 \}$$

Under the measure $\mathbb{P}^*$ given by $\mathbb{P}^*(A) = \int_A Z(\omega)d\mathbb{P}(\omega)$ where $Z = e^{-\frac{\alpha^2 t}{2} + \alpha W_t}$, Girsanov’s theorem implies that $\tau_\alpha$ has distribution $F$. Then the equality $\mathbb{E}_{P^*}(1_{\tau_\alpha \leq t}) = \mathbb{E}_P(1_{\tau_\alpha \leq t} Z)$
becomes

\[ F(t) = \int_0^t \mathbb{E}_P(e^{-\frac{\alpha^2 t}{2} + \alpha W_t} | \tau_\alpha = s) F_\alpha(ds) = \int_0^t \mathbb{E}_P(e^{\alpha W_\tau} e^{-\frac{\alpha^2 t}{2} + \alpha(W_t-W_s)} | \tau_\alpha = s) F_\alpha(ds) \]

\[ = \int_0^t e^{\alpha(b(s)+\alpha s)} e^{-\frac{\alpha^2 t}{2} + \frac{\alpha^2(t-s)}{2}} F_\alpha(ds) = \int_0^t e^{\alpha b(s)+\frac{\alpha^2 s}{2}} F_\alpha(ds) \]

where we have used the almost sure equality \( W_{\tau_\alpha} = b(\tau_\alpha) + \alpha \tau_\alpha \). Since the above is true for all \( t \geq 0 \) we have

\[ F_\alpha(dt) = F(dt)e^{-\alpha b(t)-\frac{\alpha^2 t}{2}} \quad (2.36) \]

Under the assumption \( b_\alpha(t) \geq c \), we know that \( \tau_\alpha \leq \tau_c \) a.s. for all \( \alpha \in \mathcal{A} \), where \( \tau_c := \inf\{t > 0; W_t \leq c\} \). Since \( \tau_c \) is almost surely finite then so is \( \tau_\alpha \), which implies \( F_\alpha(\infty) = 1 \). Thus, for \( \alpha \in \mathcal{A}_{b(t)} \), integrating (2.36), we obtain the Fredholm integral equation of the first kind

\[ \int_0^\infty e^{-\alpha b(s)-\frac{\alpha^2 s}{2}} F(ds) = 1 \quad (2.37) \]

with kernel \( K(\alpha,s) = e^{-\alpha b(s)-\frac{\alpha^2 s}{2}} \). Note that equation (2.36) holds for any continuous boundary \( b \) and \( \alpha \in \mathbb{R} \), while equation (2.37) holds for any \( \alpha \in \mathcal{A}_{b(t)} \).

Equation (2.37) can also be derived using the martingale property of the Geometric Brownian motion together with the optional sampling theorem. It appears previously in the FPT related literature and has been found as early as Shepp (1967). More recently, the equation was used in Daniels (2000) to derive expansions for the density function using perturbations of the boundary. However, to our knowledge, the extension of (2.37) for complex values of \( \alpha \) has not been undertaken so far in the context of the FPT problem. Such an extension is presented next.
Consider the processes

\[ X_t = e^{-xW_t + \frac{y^2 - x^2}{2}t} \cos(yW_t + xyt) \]
\[ Y_t = e^{-xW_t + \frac{y^2 - x^2}{2}t} \sin(yW_t + xyt) \]

for \( x, y \in \mathbb{R} \). Both processes are martingales for all real \( x, y \) and thus the process \( Z_t = X_t - iY_t = e^{-\alpha W_t - \frac{\alpha^2}{2}t} \) is a complex valued martingale where \( \alpha = x + iy \). Define the class \( \mathcal{B} \) of continuous functions \( b \).

**Definition 2** Denote the class of functions \( \mathcal{B} \subset \mathbb{C}^0([0, \infty)) \) such that \( b(.) \in \mathcal{B} \) implies that there exists a constant \( c < 0 \) such that for all \( u < 0 \), \( b(t) + ut > c \) for large enough \( t \).

Note that \( b(t) \in \mathcal{B} \) implies \( b(t) \) is uniformly bounded below and thus the corresponding first passage time is almost surely finite.

**Theorem 6** If \( b \in \mathcal{B} \) and is continuous on \([0, \infty)\), then for all complex \( \alpha \) with \( |\arg(\alpha)| \leq \pi/2 \), one has

\[
\int_0^\infty e^{-\alpha b(s) - \frac{\alpha^2}{2}s} F(ds) = 1 \quad (2.38)
\]

**Proof.** First notice that for \( b \in \mathcal{B} \) equation (2.37) holds for all real \( \alpha \) since \( b(t) + \alpha t \geq b(t) - |\alpha|t > c \) for \( t \) large enough. We first look at the quantity \( \mathbb{E}(e^{r\tau/2}) \), \( r > 0 \). For any such \( r \), since \( b \in \mathcal{B} \), there exists an \( 0 < N(r) < \infty \) and a \( c \in \mathbb{R} \) such that for \( t > N(r) \) we have \( \sqrt{r}b(t) > \sqrt{r}(c + \sqrt{r}t) = rt + c\sqrt{r} \). Then

\[
\mathbb{E}(e^{\frac{r}{2}\tau}) = \mathbb{E}(e^{\frac{r}{2}\tau}1(\tau \leq N(r))) + \mathbb{E}(e^{\frac{r}{2}\tau}1(\tau > N(r))) \leq e^{\frac{r}{2}N(r)} + e^{-c\sqrt{r}} \int_{N(r)}^\infty e^{\sqrt{r}b(t) - rt/2} F(dt) \leq e^{\frac{r}{2}N(r)} + e^{-c\sqrt{r}} \int_0^\infty e^{\sqrt{r}b(t) - rt/2} F(dt) < \infty
\]
Next we apply the optional sampling theorem by showing \( \mathbb{E}(|Z_\tau|) < \infty \) and \( \lim_{t \to \infty} \mathbb{E}(Z_1 1_{\tau > t}) = 0 \). For \( x \geq 0 \) and using the finiteness of \( \mathbb{E}(e^{r\tau/2}), \ r > 0 \), we have

\[
\mathbb{E}(|X_\tau|) \leq \mathbb{E}(e^{-xb(r)+\frac{y^2-x^2}{2} \tau}) \leq e^{-xc'} \mathbb{E}(e^{\frac{y^2-x^2}{2} \tau}) \leq \infty.
\]

where \( c' \) is the uniform lower bound of \( b \). Similarly, for \( x > 0 \), we can find an \( M(x,y) \) such that for \( t > M(x,y) \) we have the inequality \( xb(t) - \frac{y^2-x^2}{2} t > 0 \). Then, for \( t > M(x,y) \) we have \( |X_t| 1(\tau > t) \leq e^{-xb(t)+\frac{y^2-x^2}{2} t} 1(\tau > t) < 1(\tau > t) \) and thus \( \lim_{t \to \infty} \mathbb{E}(|X_t| 1_{\tau > t}) = \lim_{t \to \infty} \mathbb{P}(\tau > t) = 0 \) since \( \tau \) is almost surely finite. For \( x = 0 \)

\[
\mathbb{E}(|X_t| 1(\tau > t)) \leq \mathbb{E}\left(e^{y^2 t/2} 1(\tau > t)\right) \leq \mathbb{E}\left(e^{y^2 t/2} 1(\tau > t)\right) = \int_0^\infty e^{y^2 s/2} F(ds) < \infty
\]

and thus \( \lim_{t \to \infty} \mathbb{E}(|X_t| 1(\tau > t)) < \lim_{t \to \infty} \int_0^\infty e^{y^2 s/2} F(ds) = 0 \). Thus, for all \( x \geq 0 \), by the optional sampling theorem, \( X_t \) and \( \tau \) satisfy \( \mathbb{E}(X_t) = X_0 = 1 \). The same arguments applied to the process \( Y_t \) yield \( \mathbb{E}(Y_t) = Y_0 = 0 \). Thus \( \mathbb{E}(Z_\tau) = \mathbb{E}(X_\tau) - i\mathbb{E}(Y_\tau) = 1 \) and the proof is completed. \( \square \)

The above result gives an extension of the Fredholm equation (2.37) for boundaries belonging to the class \( \mathcal{B} \). When \( -\frac{\pi}{4} \leq \text{arg}(\alpha) \leq \frac{\pi}{4} \) it is sufficient that \( b(t) \) is uniformly bounded below for equation (2.38) to hold since \( y^2 - x^2 < 0 \).

Fredholm equations of the first kind are notoriously difficult to solve (even in the case when there is a unique solution). The two general cases in which explicit results are available are equations with kernels of the form \( K(\alpha t) \) or \( K(\alpha - t) \). In the first case we can obtain the Mellin transform of the solution and in the second the Laplace transform. Next we examine boundaries which give rise to such kernels. The following results for the linear, square-root and quadratic boundaries are well known, however, here we demonstrate that they all follow from equations (2.37) and (2.38) and illustrate their importance. Furthermore, the alternative approach presented below unifies the derivation of the analytical results for these
three best known classes of boundaries.

**Example (linear boundary):** \( b(t) = -a + bt, \ a > 0, b > 0 \).

Thus \( A = \{\alpha \geq -b\} \) and equation (2.37) becomes

\[
\int_0^\infty e^{-\left(\frac{\alpha^2}{2} + \alpha b\right)t} f(t) dt = e^{-\alpha a}
\]

If \( \tilde{f} \) is the Laplace transform of \( f \) then the above equation reads \( \tilde{f}(u) = e^{-b + \sqrt{b^2 + 2u}} \) and this is the Laplace transform of the well known Bachelier-Levy formula:

\[
f(t) = \frac{a}{\sqrt{2\pi t^{3/2}}} e^{-\left(a - bt\right)^2/(2t)}
\]

**Example (square-root boundary):** \( b(t) = p\sqrt{t} - q, \ q \geq 0, \ p \neq 0 \).

Then \( A = \{\alpha \geq 0\} \) and equation (2.37) becomes

\[
\int_0^\infty e^{-\alpha p\sqrt{t} - \frac{\alpha^2}{2} t} f(t) dt = e^{-\alpha q}
\]

Multiplying both sides of the above equation by \( \alpha x^{-1}, \ x > 0 \) and integrating \( \alpha \) on \([0, \infty)\) we obtain

\[
\int_0^\infty \alpha x^{-1} \int_0^\infty e^{-\alpha p\sqrt{t} - \frac{\alpha^2}{2} t} f(t) dtd\alpha = \int_0^\infty \alpha x^{-1} e^{-\alpha q} d\alpha
\]

\[
\Rightarrow \int_0^\infty f(t) dt \int_0^\infty \alpha x^{-1} e^{-\alpha p\sqrt{t} - \frac{\alpha^2}{2} t} d\alpha = \frac{\Gamma(x)}{q^x}
\]

\[
\Rightarrow \int_0^\infty f(t) \left(\frac{2}{2}\right)^{-\frac{x}{2}} \Gamma(x) e^{p^2/4 D_{-x}(p)} dt = \frac{\Gamma(x)}{q^x}
\]

\[
\Rightarrow \int_0^\infty t^{-\frac{x}{2}} f(t) dt = \frac{e^{-\frac{p^2}{4} q^{-x}}}{D_{-x}(p)}
\]

where \( D \) is the parabolic cylinder function. The last equality gives us the Mellin transform of \( f \) if we replace \( x \) with \( 2(1 - x), \ x < 1 \). Alternatively, by making the substitution \( t = e^u \)
in the last equation we obtain

$$\int_{-\infty}^{\infty} e^{-(\frac{x}{2}-1)u} f(e^u) du = \frac{e^{-\frac{\rho^2}{4} q^{-x}}}{D_x(p)}$$

which gives us $\tilde{f}(x)$, the Laplace transform of $f(e^u)$, after replacing $x$ with $2x + 2$

$$\tilde{f}(x) = \frac{e^{-\frac{\rho^2}{4} q^{-(2x+2)}}}{D_{-(2x+2)}(p)}, \quad x > -1 \quad (2.39)$$

A similar approach was used in Shepp (1967) for the first-passage time to the double boundary $\pm a\sqrt{t} + b + c$, $a, b, c > 0$. Novikov (1981) generalizes this result to stable processes with Laplace transform $E(\exp(\lambda X_t)) = \exp(d\lambda^\alpha t)$, $d > 0$, $\lambda \geq 0$, $1 < \alpha \leq 2$, and the boundary $b(t) = a(t + b)^{-1/\alpha} + c$. He uses the optional sampling theorem applied to the martingale $(t + b)^v H(v, \alpha, (X_t - c)(t + b)^{-1/\alpha})$ where $H(v, \alpha, x) = \int_0^\infty y^{-\alpha v - 1} \exp(xy - y^\alpha / \alpha) / \Gamma(-\alpha v) dy$.

Note that the case $\alpha = 2$ corresponds to the FPT of the Brownian motion to the square-root boundary and $H(v, 2, x) = \exp(-x^2/4) D_{2v}(x)$. The FPT problem for the Brownian motion and the square-root boundary is equivalent to the FPT problem for the O-U process and the constant boundary for which the Laplace transform of the FPT density was obtained by Bellman and Harris (1951). The equivalence follows from the space/time change transformation which reduces the O-U process to a Brownian motion. Because of this connection many authors have used it in an attempt to derive explicitly the FPT density for the O-U process and constant boundary and thus for the Brownian motion and square-root boundary.

Breiman (1966) derives a semi-closed form for the truncated density at $t = 1$. A similar result was also obtained by Uchiyama (1980). Of particular importance are the results of Ricciardi et al. (1984) who derive an infinite series representation of the FPT density of the O-U process and constant boundary and gave the corresponding infinite series for the FPT density of the Brownian motion and the square-root boundary. The series representation was obtained using standard fixed point approach in the theory of Volterra integral equations.
Subsequently another series representation was obtained by Novikov et al. (1999) using standard analytical techniques and the entire property of the parabolic cylinder function in the Mellin transform above.

**Example (quadratic boundary):** \( b(t) = \frac{pt^2}{2} - q, \ p, q > 0. \)

Take \( \alpha \) such that \( \Re(\alpha) > 0. \) Denote \( \alpha' = \alpha(2p)^{1/3} \) then \( \Re(\alpha) > 0 \) as well. Using \( \alpha' \) equation (2.38) becomes

\[
\int_0^\infty e^{-\alpha'\frac{t^2}{2}} s f(s) ds = e^{-\alpha'q}
\]

and after completing the cube under the integral, multiplying both sides of the equation by \( e^{\frac{\alpha'\beta}{2\pi i}}, \ \beta > 0 \) and integrating \( \alpha \) over any contour \( C(\alpha) \) with end points \( \infty e^{-i\frac{\pi}{3}} \) and \( \infty e^{i\frac{\pi}{3}} \) and \( |\arg(\alpha)| \leq \pi/3 \) (see Figure 2.1), we obtain

\[
\int_C e^{\beta(2p)^{1/3}(\alpha + \frac{pt}{(2p)^{1/3}})} e^{-\frac{1}{3}(\alpha + \frac{pt}{(2p)^{1/3}})^3} \alpha d\alpha = \int_C e^{\alpha(2p)^{1/3}(\beta - q) - \frac{\alpha^3}{3}} \alpha d\alpha
\]

The right hand side of the last equation is \( Ai((2p)^{1/3}(\beta - q)) \), where \( Ai \) is the Airy function (see (A.15)). Next we examine the contour integral on the left side. Define the contour \( C' = C + \frac{pt}{(2p)^{1/3}} \) and let \( z_1, z_2 \) be points on \( C \) and their corresponding images on \( C' \) be \( z'_1 \) and \( z'_2 \) (see Figure 2.1). Since the function under the contour integral is analytic, its integral over the simple closed contour \( z_1z'_1z'_2z_2 \) is 0. Thus, sending \( z_1 \to \infty e^{i\frac{\pi}{3}} \) and \( z_2 \to \infty e^{-i\frac{\pi}{3}} \) we obtain

\[
\int_C e^{\beta(2p)^{1/3}(\alpha + \frac{pt}{(2p)^{1/3}})} e^{-\frac{1}{3}(\alpha + \frac{pt}{(2p)^{1/3}})^3} \alpha d\alpha = \int_{C'} e^{\beta(2p)^{1/3}\alpha} e^{-\frac{1}{3}\alpha^3} d\alpha = Ai(\beta(2p)^{1/3})
\]

since the contributions on the legs \( z_1z'_1 \) and \( z_2z'_2 \) diminish in the limit. Therefore the Laplace
transform of $e^{\frac{t^3}{6\sigma^2}}f(t)$ is given by

$$
\psi(\sigma) := \int_0^\infty e^{-\sigma t} e^{\frac{t^3}{6\sigma^2}}f(t)\,dt = \frac{Ai\left(\frac{2^{1/3}}{\sigma^{2/3}}(\sigma - pq)\right)}{Ai(\sigma^{2/3})} \quad (2.40)
$$

This result was first obtained by Salminen (1988) using a change of measure to rewrite the survival function $\mathbb{P}(\tau > t)$ as a conditional expectation of a functional of the Brownian motion. He then evaluates this conditional expectation as a limit of the solution to a certain boundary and initial value problem and obtains an infinite series representation of the FPT density in terms of the Airy function and its zeros on the negative half-line. The same result was obtained independently by Groeneboom (1989) using a factorization of the density $f(t)$, involving a Bessel bridge and a killed Brownian motion.

![Figure 2.1: The contours of integration for quadratic boundaries in Example 3.](image)

In general we can obtain Fredholm equations of the first kind through a multiplication of equation (2.37) by a function $v(\beta, \alpha)$ (here $\beta$ can be a vector of parameters) and integrating out $\alpha$ (assuming we can exchange the two integrals). Such a transformation results in a Fredholm equation of the first kind with a new kernel function. The new equation has the form $K^*.f = \int_C v(\beta, \alpha)\,d\alpha$ where $C \subset \mathcal{A}_{b(t)}$ and $K^*$ is the new operator with kernel $K^*(\beta, t)$. 
There is one function in particular for which the new kernel has an explicit form. Let us multiply equation (2.37) by

\[ v(z, y, p, \alpha) = e^{-z\alpha - y\alpha^2/2} \alpha^{p-1}, \quad p, y > 0 \]

and integrating out \( \alpha \) on the interval \([0, \infty)\) (assuming the equation holds for all such \( \alpha \)) to obtain the new Fredholm equation:

\[
\int_0^\infty (t + y)^{-p/2} e^{(z+b(t))^2/(4(t+y))} D_{-p} \left( \frac{z + b(t)}{\sqrt{t+y}} \right) f(t) dt = y^{-p/2} e^{\frac{(a+z)^2}{4y}} D_{-p} \left( \frac{a + z}{\sqrt{y}} \right)
\]

(2.41)

where \( a \) is the initial starting point of \( W_t \). For \( y = 0 \) and \( z > 0 \) the right hand side of (2.41) becomes \( \frac{\Gamma(p)}{a+z} \) while the corresponding kernel is as in (2.41) with \( y = 0 \). If \( z = b(y) \) and \( y > 0 \) then the kernel of (2.41) becomes symmetric. The system (2.41) is essentially the Mellin transform of equation (2.37) and thus the number of solutions of these two equations is the same.

Finally we discuss the connection between the Volterra integral equations (2.7) (with \( p < 1 \)) and the Fredholm integral equation (2.37) for a certain class of boundaries. Let \( b(t) \) be continuous and uniformly bounded below, i.e. there exists a constant \( c < 0 \) such that for all \( t \geq 0 \), \( b(t) > c \). Such boundaries satisfy (2.37) for all \( \alpha \geq 0 \). Set \( y \leq c < 0 \) and \( \alpha = \sqrt{2\beta}, \beta \geq 0 \). Then, multiplying both sides of (2.37) by \( \sqrt{\pi} 2^{p+1/2} \beta^p e^{y\sqrt{2\beta}} \), \( p < 0 \) we obtain the equation:

\[
\int_0^\infty e^{-\beta s} \sqrt{\pi} 2^{p+1/2} \beta^p e^{-\sqrt{2\beta} (b(s) - y)} F(ds) = \sqrt{\pi} 2^{p+1/2} \beta^p e^{y\sqrt{2\beta}}
\]

(2.42)

For any \(-p, z, \beta > 0\), we have, (see Gradshteyn and Ryzhik (2000), 7.728),

\[
\int_0^\infty \frac{e^{-\beta s} x^{-(p+1)}}{\sqrt{\pi} 2^{p+1/2}} e^{-\frac{z^2}{4x}} D_{2p+1}(z/\sqrt{x}) dx = \beta^p e^{-z\sqrt{2\beta}}
\]
and therefore the integral equation (2.42) can be written as:

\[
\int_0^\infty \int_s^\infty e^{-\beta t} e^{-\frac{(b(s)-y)^2}{4(t-s)}} D_{2p+1} \left( \frac{b(s)-y}{\sqrt{t-s}} \right) dt F(ds) = \int_0^\infty e^{-\beta t} e^{-\frac{y^2}{4t}} D_{2p+1} \left( \frac{-y}{\sqrt{t}} \right) dt
\]

\[
\int_0^\infty e^{-\beta t} \left[ \int_0^t e^{-\frac{(b(s)-y)^2}{4(t-s)}} D_{2p+1} \left( \frac{b(s)-y}{\sqrt{t-s}} \right) F(ds) \right] dt = \int_0^\infty e^{-\beta t} \left[ \frac{e^{-\frac{y^2}{4t}} D_{2p+1} \left( \frac{-y}{\sqrt{t}} \right)}{t^{p+1}} \right] dt
\]

Thus, for all \(0 \leq s \leq t\), \(p < 1\), \(y \leq c\), invoking the uniqueness of the Laplace transform, allows us to identify the terms in the square brackets and results in the class of integral equations

\[
\int_0^t e^{-\frac{(b(s)-y)^2}{4(t-s)}} D_{p} \left( \frac{b(s)-y}{\sqrt{t-s}} \right) F(ds) = \frac{e^{-\frac{y^2}{4t}} D_{2p+1} \left( \frac{-y}{\sqrt{t}} \right)}{t^{p+1}}, \quad p < 1 \quad (2.43)
\]

where we have substituted \(2p+1\) with \(p\). This is precisely the class of equations \(A_p(y, t), \quad p < 1, y \leq c\), for boundaries \(b(t) > c\), \(t \geq 0\)
Chapter 3

Randomized FPT

In this chapter we examine a problem related to both the first-passage and inverse passage time problems. It was originally formulated by Jackson et al. (2008) as following. Let \( \{\Omega, \mathbb{P}, \mathbb{F}\} \) denote a complete probability space and \( \mathbb{F} = \{F_t\}_{0 \leq t \leq T} \) denotes the natural filtration generated by the standard Brownian motion \( W_t \) and a random variable \( X \geq b(0) \).

We will assume that \( X \) spans the \( \sigma \)-algebra \( F_0 \) and is independent of the Brownian motion. Define

\[
\tau_X = \inf \{ t > 0; W_t + X \leq b(t) \}
\]

Without loss of generality we can assume \( b(0) = 0 \) and that \( X \) is non-negative (see Figure 3.1). If \( b(0) \neq 0 \) then we take \( X - b(0) \) as the random starting point. Furthermore we will assume that \( b(t) \) is continuous and such that \( \tau|X = x \) has a continuous density function \( f(t|x) \) (see Peskir (2002b) for sufficient conditions on the boundary).

**Definition 3 (Randomized First Passage Time Problem)** Given a boundary function \( b : [0, \infty) \to \mathbb{R} \) and a probability measure \( \mu \) on \( [0, \infty) \) find a random variable \( X \) such that the law of the randomized first passage time \( \tau_X \) is given by \( \mu \).
Throughout the chapter we will assume that the random variable $X$ exists and its distribution has a probability density function denoted by $g$. That is, we seek a density function $g$ which solves the Fredholm equation:

$$
\int_{0}^{\infty} f(t|x)g(x)dx = f(t) 
$$

(3.1)

where $f(t|x)$ is the conditional distribution of $\tau_X$ and acts as the kernel of the above equation. Generally we will look for solutions to (3.1) in the class of functions $G := \{g; \int_{R^+} |g| < \infty \text{ or } |g| < K\}$ but the focus is on the class of density functions on the positive real line which is included in $G$. Note that the boundary $b(t)$ defines $f(t|x)$ uniquely. Furthermore, any $x > 0 = b(0)$ implies that $f(0|x) = 0$ for a class of boundaries with a monotone behavior near 0 (see Peskir (2002a)) and thus a necessary condition on $f$, for such boundaries, is that $f(0) = 0$ provided that we can exchange the limit $t \downarrow 0$ and the integral in (3.1).

When $X$ is non-random, the distribution of $\tau|X, f(t|X)$, is generally not known in explicit form with the exception of the few standard cases some of which we already discussed in the previous chapters. However randomizing the sample path of the Brownian motion allows the distribution of the hitting time to probe a much wider class of distributions and some of
which may be computable analytically. Furthermore, the idea of randomizing the starting point of the process allows us to bypass both the first passage time and inverse first passage time problems by assuming the pair \((b, f)\) as given while trying to match it with a density function \(g\). If we attack the problem by looking for a solution to (3.1) it may seem, at a first glance, that this could be a formidable task since the kernel of this Fredholm equation, \(f(t|x)\), is unknown for most boundary functions. However, as we will discover below, the problem is much simpler than both the FPT and IFPT problems and allows us to obtain analytical and semi-analytical results for certain transforms of the matching distribution. Moreover, in some settings, randomizing the starting point has practical interpretation. For example in modeling mortality of a cohort we can interpret \(X\) as the initial health level of an individual. This application will be discussed in more detail in the next chapter.

We will approach the randomized FPT problem with the tools developed in the previous chapter. In other words, we will use the already discussed Volterra and Fredholm equations to obtain unique integral transforms for the function \(g\).

### 3.1 Uniqueness and Existence

We start the section by assuming the existense of a function \(g\) which solves 3.1 and using the integral equations of the previous chapter we proceed to show its uniqueness under certain conditions on the boundary \(b\). In addition, the integral equations provide us with a way to compute \(g\) analytically for a class of boundary functions. At the end of the section we will address the question of existence of the random variable \(X\).

Recall the Volterra equation of first kind (2.7), which in the context of the randomized FPT (conditional on \(X = x\)) reads:

\[
\frac{e^{-y^2/p}}{t^{(p+1)/2}} = \int_0^t e^{-\frac{(b(s) - x - y)^2}{4(t-s)}} \frac{D_p((b(s) - x - y)/\sqrt{t-s})}{(t-s)^{(p+1)/2}} f(s|x) ds
\]  (3.2)
where \(f(s|x)\) is the conditional density of \(\tau_X\) and \(y < b(t) - x\) with \(p \in \mathbb{R}\). Exploring the connection between the parabolic cylinder function and the Hermite polynomials, which form a complete orthogonal basis in \(L^2(-\infty, \infty)\) with respect to the standard normal distribution, we can derive an unique series representation of the density function \(g\) whenever it exists.

This gives us the following result:

**Lemma 3** Suppose \(b(.)\) is a continuous function on \((0, \infty)\) which takes both positive and negative values. Then, if (3.1) has a solution \(g \in G \cap L^2(e^{-u^2})\), it is unique and it is given by

\[
g(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} t^{n/2} a_n(t) \frac{H_n(x/\sqrt{2t})}{2^n n!}
\]

for any \(t > 0\) such that \(b(t) > 0\) where \(a_n(t)\) is given by

\[
a_n(t) := \int_0^t e^{-\frac{(b(s))^2}{2(t-s)}} \frac{H_n((b(s))/\sqrt{2(t-s)})}{(t-s)^{(n+1)/2}} f(s) ds
\]

**Proof.** Suppose \(b(t) > 0\) for some \(t > 0\) so that for \(y = -x\) the condition \(y = -x < b(t) - x\) is satisfied. Recall that \(D_n(u) = 2^{-n/2}e^{-u^2/4}H_n(u/\sqrt{2})\) where \(H_n\) is the Hermite polynomial of degree \(n\). Then (3.2), for \(y = -x\) and \(p = n\) reads:

\[
e^{-\frac{x^2}{2t}} \frac{H_n(x/\sqrt{2t})}{t^{(n+1)/2}} = \int_0^t e^{-\frac{(b(s))^2}{2(t-s)}} \frac{H_n((b(s))/\sqrt{2(t-s)})}{(t-s)^{(n+1)/2}} f(s|x) ds
\]

(3.4)

Multiply (3.4) by \(g(x) \in G\), assuming that there exists such a \(g\) which solves (3.1), and integrate \(x\) on \((0, \infty)\) to obtain:

\[
\int_0^\infty e^{-\frac{x^2}{2t}} \frac{H_n(x/\sqrt{2t})}{t^{(n+1)/2}} g(x) dx = \int_0^\infty \int_0^t e^{-\frac{(b(s))^2}{2(t-s)}} \frac{H_n((b(s))/\sqrt{2(t-s)})}{(t-s)^{(n+1)/2}} f(s|x) g(x) dx ds
\]

(3.5)

Next we examine the right hand side of (3.5) and justify the exchange of the order of
integration by the use of Fubini’s theorem after we show the quantity under the double integral is absolutely integrable. Thus, we have:

\[
\int_0^\infty \int_0^t e^{-(b(s))^2 \frac{(b(s))}{2(t-s)}} \frac{H_n(\frac{b(s)}{\sqrt{2(t-s)}})}{(t-s)^{(n+1)/2}} f(s|x) |g(x)| dsdx \\
\leq z_n' \int_0^\infty \int_0^t e^{-(b(s))^2 \frac{(b(s))}{2(t-s)}} \frac{q_n - \frac{b(s)}{\sqrt{2(t-s)}}}{(t-s)^{(n+1)/2}} f(s|x) |g(x)| dsdx \\
= z_n' \int_0^\infty \int_0^t e^{-(b(s))^2 \frac{(b(s))}{\sqrt{2(t-s)}} - q_n^2}{(t-s)^{(n+1)/2}} f(s|x) |g(x)| dsdx
\]

where \( q_n = \sqrt{2 \frac{[n/2]}{2}} \) and \( z_n = 2^{n/2-[n/2]}(n!/[n/2]!) \) and \( z'_n = 2^{n/2}(n!/[n/2]!) \). Let \( p_n(s, t) = e^{\frac{1}{2} \frac{(b(s))^2}{\sqrt{2(t-s)}} - q_n^2}{(t-s)^{(n+1)/2}} \). We have \( 0 \leq p_n(0, t) < t^{-(n+1)/2} \) and \( p_n(t, t) = 0 \) since \( b \) is continuous. Thus \( p_n(\cdot, s) \) is continuous and since it is finite at 0 and at \( t \) then it is bounded on the interval \([0, t]\) by, say, \( M_n(t) \). Then, continuing from (3.8) we have:

\[
z_n' \int_0^\infty \int_0^t e^{-(b(s))^2 \frac{(b(s))}{\sqrt{2(t-s)}} - q_n^2}{(t-s)^{(n+1)/2}} f(s|x) |g(x)| dsdx \leq z_n' M_n(t) \int_0^\infty \int_0^t f(s|x) |g(x)| dsdx \\
= z_n' M_n(t) \int_0^\infty F(t|x) |g(x)| dx
\]

where \( F(t|x) \) is the conditional cdf. Since \( b \) is continuous, for each \( t > 0 \), we can find an \( k(t) < 0 \) and \( N(t) \geq -k(t) \) such that \( b(s) - x < -k(t) - x < 0 \) for all \( s \leq t \) and \( x \geq N(t) \). Therefore, for \( x \geq N(t) \), \( F(t|x) \leq 2\Phi \left( -\frac{x+k(t)}{\sqrt{t}} \right) \leq \frac{\sqrt{t}e^{-(x+k(t))^2/(2t)}}{x+k(t)} \) and continuing from the last equality we obtain:

\[
z_n' M_n(t) \int_0^\infty F(t|x) |g(x)| dx = z_n' M_n(t) \left[ \int_0^{N(t)} F(t|x) |g(x)| dx + \int_{N(t)}^\infty F(t|x) |g(x)| dx \right] \\
\leq z_n' M_n(t) \left[ \int_0^{N(t)} |g(x)| dx + \sqrt{t} \int_{N(t)}^\infty e^{-(x+k(t))^2/(2t)} x + k(t) |g(x)| dx \right]
\]

The integrals in the square brackets are both finite for all \( g \in G \) (and more generally for all
3 Randomized FPT

$g \sim O(e^{x^r}), 0 \leq r < 2$ for large $x$ and which are absolutely integrable in the neighborhood of zero) and hence we can exchange the order of integration and the right side of (3.5) becomes $a_n(t)$. Furthermore, the left side of (3.5) can be written as

$$\frac{\sqrt{2}}{t^{n/2}} \int_0^\infty e^{-u^2} H_n(u)g(u\sqrt{2t})du$$

which is the Hermite transform of $g(u\sqrt{2t})$. Thus, since the Hermite functions form a complete orthogonal basis in the space $L^2(e^{-u^2})$, if the Fredholm equation

$$\int_0^\infty f(t|x)g(x)dx = f(t)$$

has a solution $g \in G \cap L^2(e^{-u^2})$, it is unique and it is given by:

$$g(x) = \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^\infty \frac{t^{n/2}a_n(t)}{2^n n!} H_n(x/\sqrt{2t})$$

provided that $b$ is continuous and $b(t) > 0$ for some $t$. □

This solution need not be a true density function. However, in the case when $g$ is a density function then it is the unique solution to the randomized FPT problem. Notice that the above series representation holds for all $t > 0$ such that $b(t) > 0$.

Let us go back to equation (3.4). When $n = 1$ and using $H_1(z) = 2z$, the equation reduces to

$$\int_0^t e^{-\frac{b(s)^2}{2t-s}} f(s|x)ds = \frac{e^{-x^2/2t}x}{t^{3/2}}$$

(3.9)

If there exists a $g \in G$ such that it solves (3.1) for the boundary $b$ and unconditional density
\[ \int_0^t e^{-\frac{b(s)^2}{2(t-s)}} b(s) ds = f_0(t) \]  

(3.10)

where \( f_b(s) = \int_0^\infty f(s|x)g(x)dx \) and \( f_0(t) = \int_0^\infty e^{-\frac{x^2}{2\sqrt{2\pi}t}} g(x)dx \). If \( b(t) > 0 \) for all \( t > 0 \) then (3.10) and (3.9) hold for all \( t > 0 \). Therefore if such a \( g \) is a density function, the corresponding unconditional distributions of the first passage times to \( b(s) \) and to 0 are related as in (3.10). Furthermore, this equation shows that for every distribution \( f_b \) for which there is a matching distribution \( g \) there exists a distribution \( f_0 \) (given by the integral in (3.10)) such that the pair \((f_0, b = 0)\) has the same matching distribution \( g \). Thus the class of unconditional densities for the boundary \( b(t) = 0 \) for which there exists a matching distribution is at least as large as the corresponding class of unconditional distributions for any boundary \( b(t) > 0 \). Furthermore, equation (3.9) has a simple probabilistic interpretation.

In order to hit 0 by time \( t > 0 \) the process \( x + W_t \) has to hit \( b \) first.

Next we examine the Laplace transform of \( g \) using the Fredholm equation (2.37) of the first kind:

\[ \int_0^\infty e^{-\alpha z(t) - \alpha t^2/2^2} h(t) dt = 1 \]

where \( h \) is the density function of the first passage time of the Brownian motion to the boundary \( z(t) \). This equation holds for all \( \alpha \) and continuous functions \( z \) satisfying \( z(t) + \alpha t > c, \forall t \geq 0 \). Using this equation we obtain the following result:

**Lemma 4** Suppose \( b(t) + \alpha t > c \) for some constant \( c \) and all \( \alpha \geq 0 \). Then, if (3.1) has a solution \( g \in G \), it is unique and its Laplace transform is given by:

\[ \tilde{g}(\alpha) = \int_0^\infty e^{-\alpha b(t) - \alpha t^2/2} f(t) dt \]  

(3.11)
Proof. Assume that equation (3.1) has a solution \( g \in G \). Under the condition \( b(t) + \alpha t > c, \forall \alpha \geq 0 \), equation (2.37) holds for all \( \alpha \geq 0 \) (conditional on \( X = x \)) and we have

\[
\int_0^\infty e^{-\alpha b(t) - ta^2/2} f(t|x) dt = e^{-\alpha x}
\]

Multiply both sides by \( g(x) \) and integrate out \( x \). By Fubini’s theorem we can exchange the order of integration since \( \int_0^\infty \int_0^\infty |g(x)| e^{-\alpha b(t) - ta^2/2} f(t|x) dtdx = \int_0^\infty e^{-\alpha x}|g(x)| dx < \infty \). Finally, using (3.1), we obtain the Laplace transform of \( g \) given in (3.11). Uniqueness follows from the uniqueness of the Laplace transform. \( \square \)

For example, suppose \( b(t) = \sqrt{t} - x \) and let \( f \) be the unconditional density. Then (3.11) becomes

\[
\tilde{g}(\alpha) = \int_0^\infty e^{-\alpha \sqrt{t} - ta^2/2} f(t) dt
\]

for all \( \alpha > 0 \). Multiplying both sides of this equation by \( \alpha^{p-1}, p > 0 \), and integrating out \( \alpha \) we obtain

\[
\int_0^\infty \alpha^{p-1} \tilde{g}(\alpha) d\alpha = \int_0^\infty \alpha^{p-1} \int_0^\infty e^{-\alpha \sqrt{t} - ta^2/2} f(t) dtd\alpha = \int_0^\infty t^{-p/2} f(t) \int_0^\infty u^{p-1} e^{-u - u^2/2} dudt = e^{1/4} \Gamma(p) D_{-p}(1) \int_0^\infty t^{-p/2} f(t) dt = e^{1/4} \Gamma(p) D_{-p}(1) \hat{f}(1 - p/2)
\]

where we used the substitution \( u = \alpha \sqrt{t} \) and \( \hat{f} \) is the Mellin transform of \( f \). Thus the Mellin transform of the Laplace transform of \( g \), denoted \( \hat{\tilde{g}} \) is given by

\[
\hat{\tilde{g}}(p) = e^{1/4} \Gamma(p) D_{-p}(1) \hat{f}(1 - p/2)
\]

provided that \( \hat{f}(1 - p/2) \) exists for a non-empty set of positive real values of \( p \).
We saw from the above lemmas that existence implies uniqueness and the question of showing the existence of a unique matching distribution reduces to finding conditions under which (3.1) has a solution. Given \( b(t) \), there may not exist a density \( g \) for every density function \( f \) satisfying (3.1). For example, let \( b(t) = 0 \) and \( f(t) = \lambda e^{-\lambda t} \), \( \lambda > 0 \). Then (3.11) reduces to \( \tilde{g}(\alpha) = \tilde{f}(\alpha^2/2) = 2\lambda/(2\lambda + \alpha^2) \). This is the Laplace transform of \( g(x; \lambda) = \sqrt{2\lambda}\sin(x\sqrt{2\lambda}) \in G \), but it is not a probability density function.

A sufficient requirement for the existence of a solution to (3.1) can be constructed based on Picard’s Criterion (see e.g. Polyandin and Manzhurov (2008), p.578-583). However, such a construction is difficult since we do not know the kernel \( f(t|x) \) explicitly with the exception of the usual class of boundaries. So we will approach the question of existence in a probabilistic way. We will look for a random variable \( X \) such that

\[
\mathbb{E}^\mu(f(t|X)) = f(t)
\]

where the expectation is taken under a measure \( \mu \) with support on the positive real line. Next we present two methods which address the question of existence of \( X \) (or equivalently \( \mu \)) such that (3.12) holds.

**Method 1**: Let \( B \) denote the class of continuous boundaries for which there exists a constant \( c < 0 \) such that for all \( u < 0 \) we have \( b(t) - x + ut > c \) whenever \( t \) is large enough and \( b(0) = 0 \). In particular we have that the FPT \( \tau|x \) for a boundary \( b \in B \) is almost surely finite for any \( x > 0 \). Furthermore, we know that for this class of boundaries and for all \( \beta \in \mathbb{R}, x \geq 0 \) we have the integral equation (2.38)

\[
\int_0^\infty e^{-i\beta b(t) + t\beta^2/2} f(t|x) dt = e^{-i\beta x}
\]
Moreover, suppose \( f \) is such that
\[
h(\alpha) := \int_0^\infty e^{-iab(t) + \frac{\alpha^2}{2} t^2} f(t) dt = \int_0^\infty K^c(\alpha, t) f(t) dt - i \int_0^\infty K^s(\alpha, t) f(t) dt
\]
is a characteristic function of a probability law \( \mu \) where \( K^c(\alpha, t) := \cos(ab(t))e^{\frac{\alpha^2}{2} t^2} \) and \( K^s(\alpha, t) := \sin(ab(t))e^{\frac{\alpha^2}{2} t^2} \). Let \( Y \) be a random variable with its law given by \( \mu \) and write \( Y \) as \( Y = Y^+ - Y^- \) where \( Y^+ = \max(Y, 0) \) and \( Y^- = \max(-Y, 0) \). Take \( \beta = \alpha, \ x = Y^+ \) and \( \beta = -\alpha, \ x = Y^- \) in (3.13) to form the two equations
\[
\int_0^\infty e^{-iab(t) + \frac{\alpha^2}{2} t^2} f(t|Y^+) dt = e^{-i\alpha Y^+} \tag{3.14}
\]
\[
\int_0^\infty e^{iab(t) + \frac{\alpha^2}{2} t^2} f(t|Y^-) dt = e^{-i\alpha Y^-} \tag{3.15}
\]
Multiplying the top equation by \( 1(Y \geq 0) \) and the second equation by \( 1(Y < 0) \) and taking expectation w.r.t. \( \mu \) we obtain
\[
\int_0^\infty e^{-iab(t) + \frac{\alpha^2}{2} t^2} f^+(t) dt = \int_0^\infty e^{-i\alpha y} \mu(dy) \tag{3.16}
\]
\[
\int_0^\infty e^{iab(t) + \frac{\alpha^2}{2} t^2} f^-(t) dt = \int_{-\infty}^0 e^{-i\alpha y} \mu(dy) \tag{3.17}
\]
where \( f^+(t) = E(f(t|Y^+)1(Y \geq 0)) \) and \( f^-(t) = E(f(t|Y^-)1(Y < 0)) \), provided that we can exchange the integral and expectation in the two equalities above. Adding the above equations we obtain
\[
h(\alpha) = \int_0^\infty e^{-iab(t) + \frac{\alpha^2}{2} t^2} f^+(t) dt + \int_0^\infty e^{iab(t) + \frac{\alpha^2}{2} t^2} f^-(t) dt
\]
\[
= \int_0^\infty K^c(\alpha, t)(f^+(t) + f^-(t)) dt - i \int_0^\infty K^s(\alpha, t)(f^+(t) - f^-(t)) dt
\]
This implies that we have to have the following two equalities which hold for all $\alpha \in \mathbb{R}$:

\[
\int_0^\infty K^c(\alpha,t)(f^+(t) + f^-(t) - f(t))dt = 0
\]

\[
\int_0^\infty K^s(\alpha,t)(f^+(t) - f^-(t) - f(t))dt = 0
\]

Now suppose that $b$ is such that if $\int_0^\infty K^c(\alpha,t)z(t)dt = 0$ for all $\alpha \in \mathbb{R}$ then $z(t) = 0$. Then we have

\[
f(t) = f^+(t) + f^-(t) = E(f(t|Y^+)1(Y \geq 0) + f(t|Y^-)1(Y < 0)) = E(f(t|Y^+ + Y^-))
\]

and taking $X = Y^+ + Y^-$ we see that the law of $X$ is the unique solution to (3.12). Furthermore, substituting $x = Y^+ + Y^-$ in (3.13) and taking expectation w.r.t. $\mu$ we obtain

\[
h(\beta) = \int_0^\infty e^{-i\beta y}\mu(dy) + \int_{-\infty}^0 e^{i\beta y}\mu(dy)
\]

But $h(\beta) = \int_{-\infty}^\infty e^{-i\beta y}\mu(dy)$ thus we have

\[
0 = \int_{-\infty}^0 (e^{i\beta y} - e^{-i\beta y})\mu(dy) = 2i\int_{-\infty}^0 \sin(\beta y)\mu(dy)
\]

Since this holds for any $\beta \in \mathbb{R}$ it has to hold $\mu(-\infty, 0) = 0$ and thus $X = Y^+$. On the other hand if $b$ is such that if $\int_0^\infty K^s(\alpha,t)z(t)dt = 0$ for all $\alpha \in \mathbb{R}$ implies $z(t) = 0$, then

\[
f(t) = f^+(t) - f^-(t) = E(f(t|Y^+)1(Y \geq 0) - f(t|Y^-)1(Y < 0))
\]

This implies that

\[
1 = \int_0^\infty f(t)dt = E(\int_0^\infty f(t|Y^+)dt1(Y \geq 0) - \int_0^\infty f(t|Y^-)dt1(Y < 0)) = P(Y \geq 0) - P(Y < 0)
\]
by Fubini’s theorem. Therefore $P(Y < 0) = 0$ and the law of $X = Y^+$ is the unique solution to (3.12). Note that for every $b(t) \in B$ and every $f$ of the form $f(t) = \int_0^\infty f(t|x)g(x)dx$ for some density $g$, the function $h(\alpha)$ is a characteristic function of positive random variable which has a distribution given by $g$.

The first disadvantage of this method is that its sufficient conditions on the existence of $X$ are difficult to check for a given boundary $b(t)$ and a density function $f$. The second disadvantage is that the restrictions on the boundary and the unconditional density are very strong. For example, in the case of the boundary, we would require that it grows faster than a linear function for large values of $t$. A less restrictive method to address the existence of the random variable $X$ is presented next.

**Method 2:** Let $b(t)$ and the unconditional density $f(t)$ be such that the function $r : [0, \infty) \to (0, \infty)$ defined as

$$r(\alpha) := \int_0^\infty e^{-ab(t) - \alpha^2 t/2} f(t) dt$$

is completely monotone in the sense that $r(\alpha)$ possesses derivatives of all orders which satisfy $(-1)^n \frac{d^n}{d\alpha^n} r(\alpha) \geq 0$ for all $\alpha > 0$ and all non-negative integers $n \geq 0$. Furthermore, suppose that $b(t)$ is such that if

$$\int_0^\infty e^{-ab(t) - \alpha^2 t/2} z(t) dt = 0 \quad (3.18)$$

for all $\alpha \geq 0$ then $z(t)$ is identically zero. The function $r$ is our candidate for a moment generating function. Since $r$ is completely monotone and $r(0) = 1$, by Bernstein’s theorem (see Feller (1971) pp. 439), there exists a probability measure on $[0, \infty)$ with cumulative distribution function $q$ such that

$$r(\alpha) = \int_0^\infty e^{-x\alpha} dq(x)$$
On the other hand if equation (2.37) holds for all $\alpha \geq 0$ (in other words there exists a $c < 0$ such that $b(t) + \alpha t \geq c$ for all $\alpha \geq 0$) then we have

$$\int_0^\infty e^{-\alpha b(t)} e^{-\alpha^2 t/2} f(t|x) dt = e^{-\alpha x}$$

for all $\alpha \geq 0$. Taking integrals on both sides of the above equation with respect to the function $q$ and using Fubini’s theorem we obtain

$$r(\alpha) = \int_0^\infty e^{-\alpha x} dq(x) = \int_0^\infty \int_0^\infty e^{-\alpha b(t)} e^{-\alpha^2 t/2} f(t|x) dt dq(x)$$

$$= \int_0^\infty e^{-\alpha b(t)} e^{-\alpha^2 t/2} \int_0^\infty f(t|x) dq(x) dt$$

The above relation implies that

$$\int_0^\infty e^{-\alpha b(t)} e^{-\alpha^2 t/2} \left( \int_0^\infty f(t|x) dq(x) - f(t) \right) dt = 0 \quad (3.19)$$

Since (3.19) holds for all $\alpha \geq 0$, together with our assumptions, we obtain

$$\int_0^\infty f(t|x) dq(x) = f(t)$$

Furthermore if we integrate the last equality w.r.t. $t$ on $[0, \infty)$ we see that $\int_0^\infty dq(x) = 1$ since $f$ is a proper density function. Therefore $q$ defines a proper distribution function and we obtain the equality $\mathbb{E}^q(f(t|X)) = f(t)$ where the distribution of $X$ is given by $q$ and so there exists a solution to the matching distribution problem (3.12) and it is given by $\mu = q$. The assumption that (3.18) has only the trivial solution is certainly satisfied in the case $b(t) = at$, $a > 0$ because, in this case, the Laplace transform of $z(t)$ is zero and thus $z(t)$ is identically zero. The assumption is also satisfied when $b(t) = a\sqrt{t}$ since, in this case, the Mellin transform of $z$ is zero (using the same approach in the derivation of the result for the
square-root boundary of Section 2.2). Thus the class of boundaries for which (3.18) has only the trivial solution is non-empty. The complete monotonicity of the function $r$ is harder to check. However, in the case when the boundary is a linear function we can compute the density function $g$ explicitly for a large class of unconditional distributions.

## 3.2 Linear Boundary

When the starting position $X$ is non-random then the first passage time distribution of the Brownian motion to the linear boundary $bt - X$ is well known and given by

$$ f_{\tau|X}(t) = \frac{X}{\sqrt{2\pi t}} e^{-\frac{(bt-x)^2}{2t}} $$

Jackson et al. (2008) use this explicit form to demonstrate that the hitting time of a drifted Brownian motion with a random starting point can replicate a Gamma distribution. In this section we corroborate this result based on our integral equation (3.11) and extend it to a class of distributions which are infinite linear combinations of Gamma distributions.

Let $b(t) = bt$, $b > 0$ and $\tau_{x,b} = \inf\{t > 0; W_t \leq bt - x\}$, $\tau_{x,0} = \inf\{t > 0; W_t \leq -x\}$, then (3.11), for $\alpha > -b$, reduces to

$$ \tilde{g}(\alpha) = \int_0^\infty e^{-t(\alpha b + \alpha^2/2)} f(t)dt = \tilde{f}(\alpha b + \alpha^2/2) $$

(3.20)

where $\tilde{f}$ is the Laplace transform of $f$, the distribution of $\tau_{X,b}$. If we can factorize $\alpha b + \alpha^2/2$ and we take $f$ to be the density of the Gamma distribution then we can write $\tilde{f}(\alpha b + \alpha^2/2)$ as a product of two Laplace transforms of Gamma densities. Thus $g$ would be a convolution of Gamma distributions. The same argument applies when $f$ is a mixture of Gamma distributions. More formally, define the sequence $a_n$ such that $\inf_{n \geq 1} a_n \geq 2/b^2$ and
the sequence \( b_n > 0, \forall n \geq 1 \). Define the class of densities,
\[
C := \left\{ f; f(t) = \sum_{n=1}^{\infty} p_n f_{a_n, b_n}(t), \sum_{n=1}^{\infty} p_n = 1, p_n \geq 0 \right\}
\]
where \( f_{a_n, b_n} \) are densities of Gamma distributions with scale parameters \( a_n \) and shape parameters \( b_n \). We saw above that for any boundary \( b(t) > 0, \forall t > 0 \), with corresponding unconditional density \( f_b(t) \), the two densities \( f_0 \) and \( f_b \) are related as in (3.10) for all \( t \). Let \( K_{b(t)} \) be the linear integral operator in (3.10) and define the class of densities \( C^{b(t)} := K_{b(t)}C \).

Then we have the following result:

**Theorem 7** Let \( b(t) = bt, b > 0 \). Suppose \( \tau_{X,b} \sim f \in C \) or \( \tau_{X,0} \sim f_0 \in C^{bt} \). Then, in both cases the matching density \( g(x) \) is the same and is given by
\[
g(x) = \sum_{n=1}^{\infty} p_n \sqrt{2\pi e^{-bx}} \left( \frac{x}{a_n \sqrt{b^2 - 2/a_n}} \right)^{b_n - 1/2} I_{b_n - 1/2}(x \sqrt{b^2 - 2/a_n}) \tag{3.21}
\]

**Proof.** First we show that 3.21 holds for a finite mixture of Gamma densities. Let \( f(t) = \sum_1^N p_n f_{a_n, b_n}(t), \sum p_n = 1, p_n \geq 0 \), where \( f_{a_n, b_n} \) are Gamma densities with scale parameter \( a_n \) and shape parameter \( b_n \). Then, from 3.20, we have
\[
\tilde{g}(\alpha) = \tilde{f}(b\alpha + \alpha^2/2) = \sum_1^N p_n (1 + a_n(b\alpha + \alpha^2/2))^{-b_n}
\]
\[
= \sum_1^N p_n (1 + c_n^+ \alpha)^{-b_n}(1 + c_n^- \alpha)^{-b_n}
\]
where \( c_n^\pm = \frac{a_n}{2}(b \pm \sqrt{b^2 - 2/a_n}) \). In order for \( c_n^\pm \) to be positive real numbers we would would require that \( a_n \geq 2/b^2, n = 1, ..., N \). From the last equality we see that \( g(x) \) is a mixture of
convolutions of Gamma r.v.’s.

\[
g(x) = \sum_{n=1}^{N} p_n \int_{0}^{x} f_{c_n^+, b_n}(u) f_{c_n^-, b_n}(x - u) du
\]

\[
= \sum_{n=1}^{N} p_n \frac{\sqrt{2\pi} e^{-bx}}{\Gamma(b_n) \sqrt{a_n}} \left( \frac{x}{a_n \sqrt{b^2 - 2/a_n}} \right)^{b_n-1/2} I_{b_n-1/2}(x \sqrt{b^2 - 2/a_n})
\]

where \(I\) is the modified Bessel function of the first kind.

For an infinite mixture of gamma distributions the result follows easily. For \(\tau_{X,b} \sim f \in C\), substitute \(f\) in (3.11) and using Fubini’s theorem we can exchange the integration and summation (since all quantities are positive) to obtain the above result. The condition \(\inf_{n \geq 1} a_n \geq 2/b^2\) ensures that each Laplace transform in the infinite mixture is factorizable with real valued roots. For the case \(\tau_{X,0} \sim f_0 \in C^b_N\), we use (3.10) (with \(b(t) = bt\)) which relates \(f \in C\) with \(f_0 \in C^b\) where \(f\) and \(f_0\) correspond to the same matching density \(g\). \(\square\)

Next we look at several simple examples for a single Gamma density:

**Example 1:** For \(N = 1\) and \(a_1 = 2/b^2\) we have \(c_1^\pm = 1/b\) and

\[
\tilde{g}(\alpha) = (1 + (1/b)(b\alpha + \alpha^2/2))^{-b_1} = (1 + \alpha/b)^{-2b_1}
\]

Thus \(g\) is the density of a Gamma\((1/b, 2b_1)\) distribution.

**Example 2:** For \(N = 1, a_1 = 2/b^2\) and \(b_1 = 1/2\) then \(g(x) = be^{-bx}\), the density of exponential distribution with parameter \(b\).

**Example 3:** For \(b_n = 1\) and \(a_n \geq 2/b^2, n \leq N\), using the equality \(I_{1/2}(u) = 2\sinh(u) \sqrt{2\pi u}\), we obtain:

\[
g(x) = 2e^{-bx} \sum_{n=1}^{N} p_n \frac{\sinh(x \sqrt{b^2 - 2/a_n})}{a_n \sqrt{b^2 - 2/a_n}}
\]

**Example 4:** When \(b > 1\) then we can take \(a_1 = 2, b_1 = k/2\) so that \(\tau_{X,b} \sim \chi^2(k)\) and \(g\)
is given by
\[
g(x) = \frac{\sqrt{\pi}e^{-bx}}{\Gamma(k/2)} \left( \frac{x}{2\sqrt{b^2 - 1}} \right)^{(k-1)/2} I_{(k-1)/2}(x\sqrt{b^2 - 1})
\]

While direct inversion of (3.20) could be complicated for a general density \( f \), the above theorem gives us a more straightforward methodology for computing densities from the class \( C \) and their corresponding matching densities given by (3.21), simply by choosing the sequences \( a_n, b_n, p_n \). Because of the restriction on the scale parameters we could set \( a_n = 1/c, c \leq b^2/2 \). Then the unconditional density function \( f \) becomes:
\[
f(t) = ce^{-ct} \sum_{1}^{\infty} \frac{p_n}{\Gamma(b_n)} (ct)^{b_n-1}
\]

This class of densities includes the non-central \( \chi^2(m) \) distribution by choosing \( b_n = m/2 + n, c = 1/2, p_n = \frac{e^{-\delta^2/2}(\delta^2/2)^{n-1}}{(n-1)!} \) where \( \delta \) is the non-central parameter. Some more general examples of distributions of the form (3.22) are given below.

**Example 5:** \( p_n = e^{-a}a^{n-1}/(n-1)! \), \( b_n = v + n, a_n = 1/c \). Then \( f \) and \( g \) are given by:
\[
f(t) = ce^{-ct-a} \sum_{k=0}^{\infty} \frac{(act)^{v+k}}{a^v k!(v+k+1)} = \frac{c^{v/2+1}t^{v/2}e^{-ct-a}}{a^{v/2}} I_v(2\sqrt{act})
\]
\[
g(x) = \sqrt{2\pi} e^{x+1} \left( \frac{x}{\sqrt{b^2 - 2c}} \right)^{v+1/2} e^{-bx-a} \sum_{k=0}^{\infty} \frac{(xca)^k}{\sqrt{b^2 - 2c}} I_{v+k+1/2}(x\sqrt{b^2 - 2c})
\]

When \( v = 0, a = \frac{\alpha^2}{2\gamma^2} \), \( c = \frac{1}{2\gamma^2} \) then \( f(t) = e^{-t}\frac{\alpha^{t/(2\beta^2)}}{2\beta^2} I_0(\frac{\alpha\sqrt{t}}{\beta^2}) \), so that if \( \tau \sim f \) then \( \sqrt{\tau} \) has Rice distribution with parameters \( (\alpha, \beta) \).

**Example 6:** Suppose \( b_n = n \) and \( p_n = \frac{(\alpha_1)_{n-1}(\alpha_r)_{n-1}}{\beta_1\beta_2...\beta_q} / K \) where \( (x)_n = x(x+1)...(x+n-1) \) is the Pochhammer symbol. Furthermore, we assume that the real valued sequences \( \{\alpha_i\}_{i=1,...,r} \) and \( \{\beta_j\}_{j=1,...,q} \) are such that \( p_n > 0 \) for all \( n \geq 1 \) and \( \sum_{1}^{\infty} p_n = K < \infty \). Then \( f \)
and \( g \) are given by

\[
    f(t) = ce^{-ct} \sum_{n=1}^{\infty} \frac{p_n}{\Gamma(b_n)} (ct)^{b_n-1} = \frac{ce^{-ct}}{K} \sum_{n=0}^{\infty} \frac{Kp_n(ct)^n}{n!} = \frac{ce^{-ct}}{K} r_F(q)(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_q; ct)
\]

\[
    g(x) = \sqrt{2\pi} ce^{-bx} \sum_{n=1}^{\infty} \frac{p_n}{(n-1)!} \left( \frac{cx}{\sqrt{b^2 - 2c}} \right)^{n-1/2} I_{n-1/2}(x\sqrt{b^2 - 2c})
\]

where \( r_F \) is the generalized hypergeometric series (see Gradshteyn and Ryzhik (2000), 9.14).

In the case when \( r = q = 1 \) then \( p_n = \frac{(\alpha)n!}{(\beta)n-1} \) and when \( \alpha > 0, \beta > \alpha + 1 \) we ensure \( p_n > 0 \) for all \( n \) and the series \( \sum_{n=1}^{\infty} p_n \) converge by Raabe’s convergence test:

\[
    \lim_{n \to \infty} n(p_n/p_{n+1} - 1) = \lim_{n \to \infty} n(\beta - \alpha)/(\alpha + n) = \beta - \alpha > 1
\]

In this case \( f \) is given by

\[
    f(t) = ce^{-ct} \sum_{n=1}^{\infty} \frac{p_n}{\Gamma(b_n)} (ct)^{b_n-1} = \frac{ce^{-ct}}{K} \sum_{n=0}^{\infty} \frac{Kp_n(ct)^n}{n!} = \frac{ce^{-ct}}{K} r_F(1, ..., 1; ct)
\]

where \( 1_F \) is the confluent hypergeometric function (see Gradshteyn and Ryzhik (2000), 9.21).

The case when \( b < 0 \) is more straightforward. In this case, if we work with the cumulative distribution functions of \( \tau_{X,b} | X = x \) and \( \tau_{X,b} \) and look for solutions \( g \in G \) then we have the Fredholm equation:

\[
    \int_0^{\infty} F_b(t|x)g(x)dx = F_b(t) \tag{3.23}
\]

where \( F_b(t|x) \) is the conditional c.d.f. of \( \tau_{X,b} | X = x \) and \( F_b(t) \) is the unconditional c.d.f. of \( \tau_{X,b} \). Since \( F_b(t|x) \) is the c.d.f. of an inverse gaussian random variable (see equation (1.11)),

\[
    F_b(t|x) = I_{n-1/2}(x\sqrt{b^2 - 2c})
\]
sending \( t \uparrow \infty \) in (3.23) and denoting \( c := 2b < 0 \) we obtain:

\[
\int_0^\infty e^{-cx} g(x) \, dx = F_{c/2}(\infty)
\]  

using the fact that \( \tau_{X,b} \mid X = x \) can be infinite with probability \( 1 - e^{2bx} \). Thus if (3.23) has a solution then it is unique and its Laplace transform is given by \( \tilde{g}(c) = F_{c/2}(\infty) \) (assuming \( g \) does not dependent on the slope \( b \)). Clearly, for this solution to be a density function, we need \( F_{c/2}(\infty) \) to be a moment generating function in \( c \). This representation of \( \tilde{g} \) is a special case of equation (3.11) with \( \alpha = -2b \), which still holds for all \( \alpha \geq -b > 0 \). Thus, again, existence implies uniqueness.

Next we investigate the distribution of \( \tau \) after a change in the slope of the linear boundary from \( b > 0 \) to \( \mu \in \mathbb{R} \) while keeping the distribution of the starting point \( X \) unchanged. This slope change would be needed in the mortality model of Section 4.2. More formally denote

\[
\tau_k = \inf\{t > 0; W_t \leq kt - X\}
\]

where \( k = b \) or \( k = \mu \) and \( X \) is the random starting point. Suppose the unconditional distribution of \( \tau_k \) is assumed to be \( f_b \) and let \( g \) be the density function of the distribution of \( X \) which matches the pair \((bt, f_b)\). That is, we assume there exists a solution to the matching distribution problem for the boundary \( bt \) and the unconditional density \( f_b \). Next, suppose we change the slope of the linear boundary to \( \mu \) while keeping the distribution of \( X \) unchanged. The question we would like to discuss next is to find the distribution of \( \tau_\mu \). Note that a change in the slope of the boundary induces a measure change. If we define the Radon-Nykodim derivative

\[
\left( \frac{d\tilde{P}}{dP} \right)_t = e^{-(\mu - b)^2 t - (\mu - b) W_t}
\]

then under the new measure \( \tilde{P} \), \( W_t + (\mu - b)t \) is a Brownian motion and \( \tau_\mu \) has distribution
given by $f_{b=\mu}$. We can try to explore the relationship between $\tilde{P}$ and $P$ in order to obtain the distribution of $\tau_\mu$, $f_{\tau_\mu}$, under $\mathbb{P}$. Another approach is to simply compute the integral in (3.1) since we know both the conditional distribution $f_{\tau_\mu}(t|X = x)$ and $g(x)$. Perhaps the most elegant approach is to use equation (3.20) which gives us the Laplace transform of $f_{\tau_\mu}$ directly.

$$\tilde{f}_{\tau_\mu}(\alpha \mu + \alpha^2/2) = \tilde{g}(\alpha) = \tilde{f}_b(\alpha b + \alpha^2/2)$$

assuming the distribution of $X$ is unchanged. In order for this equality to hold we need the restriction $\alpha > \max(-b, -\mu)$ so that equation (3.11) holds for both linear boundaries $bt$ and $\mu t$. Reparametrizing (3.25) we get the Laplace transform of $f_{\tau_\mu}$ given by:

$$\tilde{f}_{\tau_\mu}(s) = \tilde{f}_b\left(\mu (\mu - b) + s + \sqrt{2}(b - \mu)\sqrt{s + \mu^2/2}\right), \ s > 0$$

(3.26)

Note that when $s > 0$ the quantity in the brackets on the right side of (3.26) is positive for all $b > 0$ and $\mu \in \mathbb{R}$ and thus the right hand side exists for any density function $f_b$. However, for $\mu < 0$ (3.26) is not a proper distribution. We can see this directly from the quantity in the brackets by taking $s = 0$. The advantage of using (3.26) to evaluate the new distribution is that we do not need explicit knowledge about $g$. All we need to know is that such a $g$ exists and it solves the matching distribution problem for the boundary $bt$ and the unconditional density $f_b$. In the case when $f_b$ is Gamma($a, v$) (and in order to ensure the existence of $g$ we would impose the condition $a > 2/b^2$) (3.26) becomes

$$\tilde{f}_{\tau_\mu}(s) = d^v \left(d + \mu (\mu - b) + s + \sqrt{2}(b - \mu)\sqrt{s + \mu^2/2}\right)^{-v}$$

(3.27)

$$= d^v \left(d + A + s + B\sqrt{s + C}\right)^{-v}$$

(3.28)

where $d = 1/a$, $A = \mu (\mu - b)$, $B = \sqrt{2}(b - \mu)$, $C = \mu^2/2$. When $b > \mu$ we can obtain
an integral representation for \( f_\mu \) which has a simpler form then the integral in (3.1) with \( g \) given by the terms of the series in (3.21). In this case, using Lemma 9, the last expression is the Laplace transform of

\[
f_\mu(t) = \frac{Ba^ve^{-tC}}{2\Gamma(v)\sqrt{\pi}} \int_0^t (t-x)^{-3/2}x^v e^{-x(-C+1/a+A)-\frac{2b^2}{4v}\pi} dx \tag{3.29}
\]

\[
= \frac{Be^{-tC}}{2\sqrt{\pi}} \int_0^t x(t-x)^{-3/2}e^{-x(-C+A)-\frac{a^2b^2}{4v}\pi} f_b(x) dx \tag{3.30}
\]

Finally, we end the section with a discussion on linear boundaries with random slope. What happens if we randomize the slope while keeping the intercept deterministic? The answer is given by the following relation

\[
\tau_{X,b} = \inf \{t > 0; X + W_t \leq bt\} =^d \inf \{t > 0; 1 + W_t/X^2 \leq Xbt/X^2\} = X^2 \inf \{u > 0; 1 + W_u \leq bXu\} = X^2 \tau_{1,bX}
\]

provided that \( X \) has no mass at 0. Thus, if we know the distribution of \( X \) then we know the distribution of \( \tau_{1,bX} =^d \tau_{X,b}/X^2 \) and the question of finding the distribution of the random intercept is equivalent the finding the distribution of the random slope. Suppose the slope \( b = a/X \) where \( a \geq 0 \) is a constant and \( X \) is the random intercept. Then, following the above time change argument, we obtain

\[
\tau_{X,a/X} = X^2 \inf \{u > 0; W_u \leq au - 1\} = X^2 \tau_{1,a} \Rightarrow \log \tau_{X,a/X} = 2 \log X + \log \tau_{1,a}
\]

and \( X \) and \( \tau_{1,a} \) are independent. Thus, given \( \log \tau_{X,a/X} \sim f \), the Laplace transform of the
density of \( \log X \), \( g \) (provided that it exists), is given by

\[
\tilde{g}(2\alpha) = \frac{\hat{f}(\alpha)}{h(\alpha)}
\]

where \( \tilde{h}(\alpha) \) is the moment generating function of the logarithm of an inverse gaussian random variable with parameters 1 and \( a \).

### 3.2.1 Back to the Classical FPT

In this final section we will attempt to motivate the use of the randomized FPT to attack the classical FPT of Chapter 2. We will work with the process \( W_t + x \) where \( x > 0 \) is the non-random starting point of the process. The basic idea is to construct an integral equation involving the conditional density \( f(t|x) \) with \( x \) acting as a free parameter and to search for solutions \( g \) of equation 3.1 such that the ‘unconditional’ density function \( f(t) \) is a kernel of a unique integral transform (e.g. exponential density). Note that, in this case, we do not need \( g \) to be a proper density function. We start with the construction of integral equations which are more appropriate for such manipulation than the equations of Chapter 2.

Let us consider any two continuous boundaries \( b_1, b_2 \) satisfying \( b_1(t) > b_2(t), \ t \in (0,T] \). For any such boundaries define the corresponding FPT’s as:

\[
\tau_i = \inf\{t > 0; x + W_t \leq b_i(t)\}, \ i = 1, 2
\]

with c.d.f.’s and probability densities \( F_{b_i}(t|x) \) and \( f_{b_i}(t|x) \) respectively. Clearly \( \tau_2 > \tau_1 \). Then, conditional on \( \tau_1 = s < T \) we have:

\[
\tau_2|\tau_1 = \inf\{t > \tau_1; x + W_t \leq b_2(t)\} = \inf\{t > \tau_1; x + W_{\tau_1} + W_t - W_{\tau_1} \leq b_2(t)\}
\]

\[
= \inf\{t > \tau_1; b_1(\tau_1) + W_t - W_{\tau_1} \leq b_2(t)\} = \inf\{u > 0; W_u \leq b_2(u + s) - b_1(s)\}
\]
where the last equality is in distribution since \( W_t - W_s = W_{t-s}, \ s < t, \) in distribution. Thus the stopping time \( \tau := \tau_2|\tau_1 \) is the first passage time of a Brownian motion to the boundary \( b_2(u+s) - b_1(s) \) for \( 0 < s < T \) given \( \tau_1 = s \). Let \( \mathbb{P}(\tau \leq y; s) \) denote the c.d.f. of \( \tau \) conditional on \( \tau_1 = s \). Therefore for any \( t < T \) we have

\[
F_{b_2}(t|x) = \mathbb{P}(\tau_2 \leq t) = \int_0^t \mathbb{P}(\tau \leq t-s; s)f_{b_1}(s|x)\,ds \tag{3.31}
\]

and the quantity \( \mathbb{P}(\tau \leq t-s; s) \) is independent of \( x \). We can suppress the notation \( '|x' \) by including \( x \) in the boundaries \( b_1, b_2 \). Also, since \( \mathbb{P}(\tau \leq t-s; s) = 0 \) for \( s \geq t \) we can write (3.31) as:

\[
F_{b_2}(t) = \int_0^\infty \mathbb{P}(\tau \leq t-s; s)f_{b_1}(s)\,ds
\]

There are two interesting cases. First, let \( b_2(t) = b(t) \) and \( b_1(t) = b(t) + \alpha t, \ \alpha > 0, \) so that \( b_1 > b_2 \) for all \( t > 0 \). We know the relationship between \( f_{b_1} \) and \( f_{b_2} \) from (2.36). Thus the last equation becomes

\[
F_b(t) = \int_0^\infty \mathbb{P}(\tau \leq t-s; s, \alpha)f_b(s)e^{-\alpha b(s)-\frac{\alpha^2 s}{2}}\,ds \tag{3.32}
\]

This equation can viewed as a Volterra version of the Fredholm equation (2.37) and in the limit \( t \uparrow \infty \) (3.32) converges to (2.37) provided that \( \tau_2 \) is almost surely finite. This convergence follows from the Dominated convergence theorem since \( \mathbb{P}(\tau \leq t-s; s, \alpha) \uparrow 1 \) as \( t \uparrow \infty \) and \( \tau_1 < \tau_2 \) is almost surely finite (since \( \tau_2 \) is). If we differentiate (3.32) w.r.t. \( t \) we obtain

\[
f_b(t) = \int_0^\infty f_\tau(t-s; s, \alpha)e^{-\alpha b(s)-\frac{\alpha^2 s}{2}}f_b(s)\,ds
\]

where \( f_\tau \) is the density of \( \tau \).

The second interesting case is when \( b_2(t) = b(t) - x \) where \( b(t) < 0 \) for \( t < T \) and
Randomized FPT

\[ b_1(t) = -x \] for all \( t \). Thus, for \( t < T \) equation (3.31) becomes

\[
F_b(t|x) = \int_0^\infty \mathbb{P} (\tau \leq t - s; s)f_0(s|x)ds \\
= \int_0^\infty \mathbb{P} (\tau \leq t - s; s)\frac{e^{-x^2/2s}x}{\sqrt{2\pi s^{3/2}}}ds
\]  

(3.33)

(3.34)

In this case, \( \tau \) can be viewed as the first passage time of \( W_u \) to the boundary \( b(u + s) \) with \( s \) acting as a parameter. Multiplying (3.34) by \( \sqrt{2\lambda \sin(x\sqrt{2\lambda})} \) and integrating \( x \) on \((0, \infty)\) we obtain

\[
\int_0^\infty F_b(t|x)\sqrt{2\lambda \sin(x\sqrt{2\lambda})}dx = \int_0^\infty \int_0^\infty \mathbb{P} (\tau \leq t - s; s)\frac{e^{-x^2/2s}x}{\sqrt{2\pi s^{3/2}}}\sqrt{2\lambda \sin(x\sqrt{2\lambda})}dsdx
\]

we can exchange the order of integration on the right hand side since

\[
\int_0^\infty \int_0^t |\sin(x\sqrt{2\lambda})|\frac{e^{-x^2/2s}x}{\sqrt{2\pi s^{3/2}}}dsdx \leq \int_0^\infty \int_0^t \frac{e^{-x^2/2s}x}{\sqrt{2\pi s^{3/2}}}dsdx
\]

\[
= \int_0^\infty u e^{-u^2/2}du \int_0^t 1/\sqrt{s}ds < \infty
\]

Finally we obtain

\[
\sqrt{2/\lambda} \int_0^\infty F_b(t|x)\sin(x\sqrt{2\lambda})dx = \int_0^\infty e^{-\lambda s}\mathbb{P} (\tau \leq t - s; s)ds
\]  

(3.35)

where we have used the solution to the matching distribution problem for the boundary \( b = 0 \) and the unconditional density \( f(t) = \lambda e^{-t} \) described above. The left side of equation (3.35) is the sine transform of \( F(t|x) \) w.r.t. \( x \) and the right side is the Laplace transform of \( \mathbb{P}(t-s;s) \) w.r.t. \( s \). Therefore, if we know the one side we know the other. In the context of the FPT problem, this result shows that for any regular boundary \( b(t) \) starting at \((0,0)\) (and thus the boundary is non-positive on some interval \((0,T]\)) the problem of finding the distribution of the FPT for the boundary \( b(t) - x, x > 0 \) is equivalent to the FPT problem
for the boundary $b(t + s), t \leq T$. This example illustrates the potential use of solutions to the matching distribution problem, together with an integral equation, to approach the FPT problem.
Chapter 4

Mortality Modeling with Randomized Diffusion

There are a number of ”real life” problems where one of quantities of interest is the time till ’default’ of a dynamical system. In many such problems there is some information on the distribution of this default time. In such cases it is natural to assume an underlying stochastic process and a distribution function which model reasonably the dynamical nature of the system and the behavior of the default time respectively. Under these assumptions the problem is translated into an IFPT problem. A major complication, however, is the lack of closed form results in the IFPT setting. Another drawback is that the system/process may be unobservable at the time of initiation. The matching distribution (MD) approach, discussed in Chapter 3, is perhaps more flexible as a theoretical framework for such ’real life’ problems. It extends the natural applicability of the IFPT framework since it incorporates a second source of randomness which could capture situations where the system is unobservable at the time of initiation. Furthermore it provides us with analytical or semi-analytical results and finally it is easier to deal with than the IFPT framework. Of the three inputs in the MD setting one has to be most careful in the selection of the default time distribution function. In
order to avoid any arbitrary choice of this distribution we would need a sufficient number of observations for the default time. This is the case when the default time is the time of death of an individual. In this section we present a model for mortality in the MD framework. First, we introduce the model and fit it to a Swedish cohort and then, after a brief introduction of the ’risk-neutral’ valuation framework, we discuss the pricing of mortality linked securities under the new mortality model.

4.1 The Model and the Fit

We assume that mortality of an individual is derived from a health level process which is positive and has expected decrease as individuals age. The initial \((t = 0)\) positive health level is assumed random, independent of the future behavior of the process, and we denote it with \(X\). Time of death occurs when the individual loses all of his/her ’health’ units. Formally, we assume that the health level \(h_t\) at time \(t\) is a drifted Brownian motion given by \(h_t = X - t + \beta W_t\) with \(\beta > 0\) parametrizing the volatility of \(h_t\). Thus, on average, the individual loses one health unit per unit of time (say a year). The individual dies the instance their health level hits zero; the stopping time

\[
\tau := \inf\{t : h_t = 0\}
\]  

(4.1)

defines the random time of death. As \(\beta \downarrow 0\), there is no variability in the individual’s health and it heads to zero at a rate of one unit per annum and hits zero at exactly \(h_0 = X\). Consequently, in this limit the distribution of \(\tau\) is equal to \(X\). Given a particular distribution for the random time of death - say \(f_\tau\) - then trivially choosing the initial health level \(X\) distributed as \(g(x) = f_\tau(x)\) will produce a hitting time that exactly matches the given distribution. In this way, the distribution of the random time of death can be translated to a distribution of the initial health level. This is a rather simple translation of the problem,
and (with $\beta = 0$) does not add any explanatory power into the model, nor does it lead to a
dynamic setting. However, with $\beta$ non-zero, the model becomes truly dynamic and obtaining
a distribution for $X$ which will match a given distribution of the hitting time $\tau$ is no longer
trivial. Furthermore, randomization of the initial health level allows the distribution of the
hitting time to probe a much wider class of distributions.

As a first step, suppose that the distribution of the initial health status is a mixture
of Dirac distributions, i.e. $g_X(x) = \sum_{n=1}^{N} p_n \delta(x - x_n)$, with $\sum_{n=1}^{N} p_n = 1$. Then the
unconditional distribution of the hitting time (and consequently the random time of death)
is a mixture of inverse Gaussians

$$f_\tau(t) = \sum_{n=1}^{N} p_n \frac{x_n}{\beta \sqrt{2\pi t^3}} \exp \left\{ -\frac{(x_n - t)^2}{2 \beta^2 t} \right\}.$$ (4.2)

using the well-known result for drifted Brownian motion from the FPT problem. By taking
$p_n = (l_n - l_{n-1}) / l_0$, where $l_x$ is the number of individuals alive at time $x$, and $x_n = n + \frac{1}{2}$,
this estimate of the distribution of the random time of death variable corresponds to a kernel estimator with inverse Gaussian kernel functions. In Figure 4.1, two such estimates of the
distribution are provided using Swedish cohort data with two levels of the health volatility.
Notice that with low volatilities the tail end is matched well, while the small lifetimes are over
weighted and the estimator is not smooth; contrarily, with larger volatilities, the estimator
is now smooth, short lifetimes are matched well, however long lifetimes are mismatched. Such
behavior is somewhat unsatisfactory, and we conclude that using a mixture of Dirac densities
for the initial starting point is a possibly valid approach but has some drawbacks. To address
this issue, we seek a continuous parametric family of distributions for the initial health level
which induces a useful class of parametric distributions for the unconditional hitting time
that can serve as a kernel estimator. To this end, we use the results of Chapter 3 on the MD
Figure 4.1: The kernel estimator of the distribution of the hitting time using Dirac measures for the randomized starting health unit.

problem for a linear boundary. Since the random time of death, \( \tau \), can be written as

\[
\tau = \inf \{ t : Y + W_t \leq \alpha t \}
\]

where \( Y\beta = X \) and \( \alpha = 1/\beta \), we can restate Corollary 7 in a form more relevant to our current framework.

**Proposition 2** Let the health level \( h_0 \) have the following mixture distribution

\[
g_X(x) = \sum_{n=1}^{\infty} p_n \overline{g}(x; a_n, b_n; \beta), \tag{4.3}
\]

\[
\overline{g}(x; a, b; \beta) = \frac{\sqrt{2\pi}}{(a_n)^{b_n} \Gamma(b_n) \beta} \left( \frac{x}{c_n} \right)^{b_n-\frac{1}{2}} e^{-x/\beta^2} I_{b_n-\frac{1}{2}} \left( \frac{c_n}{\beta^2} x \right), \tag{4.4}
\]

where \( I_{\nu}(z) \) is the modified Bessel function of the first kind, \( c_n = \sqrt{1-2\beta^2/a_n}, \sum_{n=1}^{\infty} p_n = 1, a_n, b_n > 0 \) and \( a_n > 2\beta^2 \forall n \). Then, the unconditional distribution of the first hitting time of
the health level is the following mixture of Gammas

\[ f_\tau(t) = \sum_{n=1}^{\infty} p_n g(t; a_n, b_n) , \quad (4.5) \]
\[ g(t; a, b) = \frac{t^{b-1} e^{-t/a}}{a^b \Gamma(b)} . \quad (4.6) \]

As we saw in the previous chapter, \( g(x; a, b; \beta) \) is the distribution of the sum of two gamma random variables with the same shape parameter but different scale parameters. By using the asymptotic form of the modified Bessel for large arguments, it is possible to show that as \( \beta \downarrow 0 \), the distribution \( g(x; a, b; \beta) \) reduces to a gamma distribution with scale \( a \) and shape \( b \). This is another way to see that as the volatility of health goes to zero, the hitting time and the initial health level have the same distribution.

Proposition 2 is a very powerful result since any distribution with positive support can be arbitrarily well approximated by a mixture of Gamma distributions (Tijms (1994)). Consequently, through randomizing the initial health level, it is possible to accurately model the time of death within the dynamic framework above.

To illustrate how this framework can be used, suppose that cohort data for the number of survivors \( l_x \) of age \( x \) is known for ages \( x = 1, \ldots, x_m \) (\( x_m \) a positive integer). Then, model the distribution of the random time of death random variable as a mixture of gamma distributions inherited from a kernel estimate of the distribution. In particular, the estimated distribution function \( \hat{f}_\tau(t) \) will be

\[ \hat{f}_\tau(t) = \sum_{x=1}^{x_m} \frac{l_x - l_{x-1}}{l_0} g \left( t; \frac{v}{x - \frac{1}{2}}, \frac{(x-\frac{1}{2})^2}{v} \right) . \quad (4.7) \]

Here, each gamma has a mean of \( x - \frac{1}{2} \) and a fixed variance of \( v \). Such a kernel estimator naturally induces smaller scale parameters as larger ages are added. In Figure 4.2, this kernel estimate to the Swedish male cohort data is illustrated. For this data set a fixed variance of
Figure 4.2: The model fit to the Sweedish cohort data. Panel (a) shows the life table data fitted with a mixture of Gamma distributions using the kernel estimator (4.7) with $v = 3^2$. Panel (b) compares the distribution of the hitting time with that of the initial level using a volatility $\beta = 0.95.\beta_{max} = 19.3\%$.

$v = 3$ was used. Given the kernel estimator (4.7), the distribution of the initial health level $g_X(x)$ which matches the given estimate of the distribution of the random time of death is provided by Corrolary 2. In Figure 4.2 panel (b), the distribution of this initial health level is shown in comparison with the distribution of the random time of death itself. To obtain the distribution of $X$, the volatility of the diffusion process $\beta$ plays the role of a free parameter; however Corrolary 2 restricts the scale parameters in relation to the volatility. There is a maximum level of variability allowed in order for the randomized diffusion to replicate a given distribution of time of death. For the kernel estimator (4.7) this maximum is

$$\beta_{max} = \sqrt{\frac{v}{2x_m - 1}}. \quad (4.8)$$

Choosing smaller variances of the gamma kernels also reduces the maximum volatility. One must trade off between fitting the target distribution very closely versus the potential to allow for more variability in the health level of individuals. In this Swedish cohort example, the maximum allowed variability of the diffusion was found to be $\beta_{max} = 20.37\%$. This is
a considerable amount of variability, yet it is interesting that the distribution of $X$ is very close to that of kernel estimate for the distribution of the time of death. As an alternative, the scale parameter can be set constant across all kernels and the shape chosen such that the mean of the kernel is equal to $x - \frac{1}{2}$. In this case the kernel estimator is

$$
\hat{f}_\tau(t) = \sum_{x=1}^{x_m} \frac{l_x - l_{x-1}}{l_0} g_{a, x - \frac{1}{2}}(t)
$$

(4.9)

and the maximum volatility is then $\beta_{max} = \sqrt{a/2}$ independent of the maximum age $x_m$. For the Swedish data set, this kernel produced similar results to the fixed variance kernel. A more parsimonious calibration procedure can be employed using the results of Willmot and Lin (2009).

### 4.2 Risk Neutral Pricing of Mortality Linked Securities

The risk-neutral pricing framework deals with the question of how to place a ‘fair-value’ on a payoff $\chi$ made at time $T > 0$, where $\chi$ is a random quantity. If $\chi$ depends on the evolution of some underlying stochastic process, the payoff is dubbed a contingent claim or a derivative in the financial literature. In particular, simple payoffs are derivatives written on a price process $S_t$, observable under a ‘real-world’ measure $P$, and are of the form $\chi = G(S_T)$ for some (non-negative) function $G$. The idea in this framework is to relate the price, $\Pi_t$ at time $t < T$, of such a claim to observable market prices of other financial instruments. Under mild conditions it is shown (see e.g. Bjork (1998)) that the price process $\Pi_t$ is given by the conditional expectation

$$
\Pi_t = \mathbb{E}_Q(e^{-\int_t^T r(s)ds} \chi | \mathcal{F}_t)
$$
where r is a risk-free interest rate and \( \mathcal{F}_t \) is a filtration generated by the R sources of randomness on the market consisting of M (traded) assets. Note that the expectation is taken under a new, equivalent to \( \mathbb{P} \), measure \( \mathbb{Q} \) called 'risk-neutral' measure under which all discounted asset prices are martingales. If \( M = R \) then the measure \( \mathbb{Q} \) is uniquely defined through the so called market price of risk while the case \( M < R \) only guarantees the existence of \( \mathbb{Q} \) (see Bjork (1998), pp 106) and the market is referred to as being incomplete. The insurance market falls into the latter category since mortality is not traded. From an actuarial point of view the risk-neutral valuation is a valid approach whenever the underlier of the contract is traded. An example of such a contract is a longevity bond which is a bond that pays a coupon that is proportional to the number of survivors in a selected birth cohort. If the underlier is not traded then the change of measure from \( \mathbb{P} \) to \( \mathbb{Q} \) is harder to justify, however, prices of many insurance contracts have an embedded premium component and as such can be viewed as 'risk-neutral' prices, computed under a measure different from the objective measure \( \mathbb{P} \).

The simplest example of a mortality linked security is a pure endowment contract with maturity \( T \) for \( x \) which pays the policyholder (of age \( x \)) $1 at time \( T \) if (s)he is still alive at that time. Using the risk-neutral valuation formula we see that the price of such a contract at time 0, denoted by \( \mathbb{E}(0, T; x) \), is given by

\[
\mathbb{E}(0, T; x) = \mathbb{E}_Q(e^{-\int_0^T r(s)ds} 1(\tau_x > T)) = \mathbb{E}_Q(e^{-\int_0^T r(s)ds}) \mathbb{Q}(\tau_x > T)
\]

where we have assumed independence between \( h_t \) and \( r(t) \). Recognizing the term \( \mathbb{E}_Q(e^{-\int_0^T r(s)ds}) \) as the risk neutral price, \( B(0, T) \), of a bond which pays $1 at time \( T \), we see that the valuation of a pure endowment is simply the product \( B(0, T) \mathbb{Q}(\tau_x > T) \). Normally, the risk neutral price of the \( T \)-bond can be determined uniquely from the prices of bonds with other maturities. The same argument applies to a longevity bond which pays annual coupons proportional to a survival index. The payments under the contract are \( 1(\tau > T_i), \ i = 1, 2, 3..., N \) at times
Thus the price of this contract at time 0, \( \mathbb{E}_{LB}(0, T; x) \), is a sum of pure endowment prices; 
\[
\mathbb{E}_{LB}(0, T; x) = \sum_{i=1}^{N} \mathbb{E}(0, T_i; x).
\]

Thus, in both cases, the computation of the prices of such contracts reduces to obtaining the survival probability \( Q(\tau > T) \) under a 'risk-neutral' measure \( Q \). Since the objective survival probability \( P(\tau_x > T) \) is observable, a common actuarial approach is to evaluate \( Q(\tau_x > T) \) based on distortion operators which transform directly the distribution of \( \tau_x \) under \( P \) to a corresponding distribution under \( Q \). Some of the most used families of such transforms are presented next (for a more detailed discussion refer to Wang (2000), Wang (1995) and Kijima and Muromachi (2008)).

1. Wang transform: The distortion operator is \( g_{\alpha}(u) := \Phi(\Phi^{-1}(u) + \alpha) \) and if \( S(t) := P(\tau > t) \) is the survival probability under \( P \) then the survival probability under \( Q_{\alpha} \) is given by \( g_{\alpha}(S(t)) \).

2. PH (proportional hazard) transform: The distortion operator is \( g_{\alpha}(u) := u^{1/\alpha} \), \( \alpha > 0 \) and the survival probability under \( Q_{\alpha} \) is given by \( g_{\alpha}(S(t)) \).

3. Esscher transform: \( Q_{\lambda}(\tau_x > t) = \frac{\mathbb{E}(1(\tau_x > t)e^{-\lambda \tau_x})}{\mathbb{E}_P(e^{-\lambda \tau_x})} \), \( \lambda > 0 \)

All of these transforms are dependent on a parameter, which could be interpreted as a market price of risk since it defines the new measure uniquely. Though these transforms possess a number of interesting and desirable properties their main disadvantage is the subjectivity of the map.

Rather than directly transforming the distribution of the time of death, under our mortality model, the measure change is induced naturally and intuitively through a change in the slope of the linear drift in \( h_t \). This slope represents the average rate of health decline and is set at \(-1\) under the measure \( P \). Thus, changing the slope, while keeping the distribution of the initial health \( X \) unchanged, results in a new distribution for the time of death \( \tau \). The slope change (and thus the measure change) simply represents our belief that under the new measure individuals’ health declines at a slower or faster rate and thus the average time of
death is lower or higher respectively.

Suppose the new slope is $-(1 + \lambda), \nonumber \lambda \neq 0$ so that the new health process becomes $h^\lambda_t := X - (1 + \lambda)t + \beta W_t$. Denoting $\hat{W}_t = W_t - \lambda t/\beta$, the new time of death is

$$\tau_\lambda := \inf\{t > 0; h^\lambda_t = 0\} = \inf\{t > 0; X - t + \beta \hat{W}_t = 0\}$$

with density function denoted by $f_{\tau_\lambda}$. Using Girsanov’s theorem we can define the new measure $Q$ such that $\hat{W}_t$ is again a Brownian motion and thus the distribution of $\tau$ under $Q$ is simply the distribution of $\tau_\lambda$ under $P$. In order to keep the new slope negative, and thus the average rate of health decline negative, we would only consider the case $1 + \lambda > 0$. Then, using (3.26), the Laplace transform of $f_{\tau_\lambda}$ (under $P$) is given by

$$\tilde{f}_{\tau_\lambda}(s) = \tilde{f}_\tau \left( s + \frac{\lambda(1 + \lambda)}{\beta^2} - \frac{\lambda}{\beta} \sqrt{2s + \frac{(1 + \lambda)^2}{\beta^2}} \right)$$

(4.10)

where $\tilde{f}_\tau$ is the moment generating function of the original time of death (prior to the slope change) defined in (4.1). For (4.10) to hold all we need to know is that there exists a random variable $X$ which matches the target density $f_\tau$. No knowledge of the distribution of $X$ is necessary. Note that (4.10) is the price (under the new distribution of the time of death) of an insurance contract which pays $1 at the time of death assuming the interest rate is a constant given by $s$. Other basic insurance products can be priced in closed form.

**Example,** Life Insurance: A contract which pays a fixed amount $F$ at time of death in exchange for a premium stream payment, $p$, during the life of the beneficiary. If we assume that the amount $p$ is payable continuously and the interest rate $r$ is a constant then $p$ is
determined from the equation:

\[ F \mathcal{E}_Q(e^{-r\tau}) = p \mathcal{E}_Q \left( \int_0^\tau e^{-rs} ds \right) \]

\[ F \tilde{f}_\tau(r) = p \frac{1 - \tilde{f}_\tau(r)}{r} \]

\[ p = rF \frac{\tilde{f}_\tau(r)}{1 - \tilde{f}_\tau(r)} \]

In the case when \( f_\tau \) is the same as in (4.7) and using (3.28), \( \tilde{f}_{\tau_\lambda} \) becomes:

\[ \tilde{f}_{\tau_\lambda}(s) = \sum_{x=1}^{x_m} \frac{l_x - l_{x-1}}{l_0} \tilde{g}(s; v, x) \tag{4.11} \]

\[ \tilde{g}(s; v, x) = \left( 1 + \frac{v}{x - 1/2} \left( \frac{\lambda(1 + \lambda)}{\beta^2} + s - \frac{\lambda}{\beta} \sqrt{2s + \frac{(1 + \lambda)^2}{\beta^2}} \right) \right)^{-(x-1/2)^2} \tag{4.12} \]

with the upper bound for \( \beta \) given by \( \beta_{\text{max}} \) in (4.8). Furthermore, in the case \( 0 < 1 + \lambda < 1 \) and using (3.30), the moment generating function in (4.11) has an explicit density function given by

\[ f_{\tau_\lambda}(t) = -\frac{\lambda e^{-t(1+\lambda)^2/2\beta^2}}{\beta \sqrt{2\pi}} \sum_{x=1}^{x_m} \frac{l_x - l_{x-1}}{l_0} \Gamma(b_x)(a_x)^{b_x} \int_0^t (t - u)^{-3/2} u^{b_x} e^{-u/a_x} e^{-u(1+\lambda)(1+3\lambda)/(2\beta^2) - \lambda^2 u^2/(2\beta^2(t-u))} du \]

\[ = -\frac{\lambda e^{-t(1+\lambda)^2/2\beta^2}}{\beta \sqrt{2\pi}} \int_0^t u(t - u)^{-3/2} e^{-u(1+\lambda)(1+3\lambda)/(2\beta^2) - \lambda^2 u^2/(2\beta^2(t-u))} \hat{f}_\tau(u) du \tag{4.13} \]

where \( \hat{f}_\tau(u) \) and the gamma density parameters \( a_x, b_x \) are given in (4.7).

Though, in the general case \( 0 < 1 + \lambda \), we do not have an explicit form for the c.d.f. of \( \tau_\lambda \) for a general density function \( f_\tau \), standard numerical techniques can be employed to invert the moment generating function in (4.10). We can, however, obtain the moments of
\( \tau_\lambda \) analytically from (4.10). For example, the first moment denoted by \( m_1^\lambda \), is given by

\[
m_1^\lambda = -\frac{d\tilde{f}_{\tau_\lambda}}{ds}\bigg|_{s=0} = -\frac{d\tilde{f}_\tau}{ds}\left(s + \frac{\lambda(1 + \lambda)}{\beta^2} - \frac{\lambda}{\beta}\sqrt{2s + \frac{(1 + \lambda)^2}{\beta^2}}\right)\left(1 - \frac{2\lambda}{2\beta\sqrt{2s + \frac{(1 + \lambda)^2}{\beta^2}}}\right)\bigg|_{s=0}
\]

\[
eq m_1\left(1 - \frac{\lambda}{1 + \lambda}\right) = \frac{m_1}{1 + \lambda}
\]

(4.15)

(4.16)

where \( m_1 = \mathbb{E}(\tau) \) is the first moment of \( \tau \), prior to the slope change. This simple result implies that the new slope, \( 1 + \lambda \), is the ratio of the two first moments; \( 1 + \lambda = \frac{m_1}{m_1^\lambda} > 0 \). Thus, not only can we change the distribution through a change in the slope but we can chose the value of the new slope in such way that the new distribution has any positive, prespecified first moment. Similarly, we can also control the second moment of \( \tau_\lambda \) using the value of the parameter \( \beta \). This, however, restricts the fitting procedure.
Chapter 5

Approximate Analytical Solutions to the FPT and IFPT Problems

In this last chapter the focus is on analytical estimation of the FPT density and boundary functions for the FPT and IFPT problems respectively.

In the FPT problem, due to the fact that we are working with linear integral equations, the natural approach to estimate the solution is the use of eigenfunctions which form a complete orthonormal basis in a normed space containing the solution. This is one of the standard approaches to approximate the solution of linear integral equations of Fredholm type. This methodology was employed in Section 5.2.

In Section 5.3 we also employ time/space change techniques to extract certain properties of the boundary in the IFPT setting and reduce the problem to finding a single boundary for a class of distributions. This boundary can be approximated analytically, with a desired precision, using the numerical approach of Chadam et al. (2006b) for solving one of the Volterra equations of Chapter 2.

We start the chapter, however, with an application of the construction developed by Lerche (1986) (outlined in Section 1.1.4) to derive a semi-closed form results for a class of
boundaries in the FPT setting. To our knowledge this class has not been explored thus far in the context of the FPT problem.

5.1 An Application of the Method of Images for a Class of Boundaries

Following the same notation as in Section 1.1.4, let us take the $\sigma$-finite measure $Q$ to be $Q(d\theta) = \theta^{p-1}d\theta; \ p > 0$ and $1/a = c^{p/2}, \ c > 0$. Then, substituting in (1.22), we obtain

$$h(x,t) = \frac{1}{\sqrt{t}} \phi(x/\sqrt{t}) - c^{p/2} \int_0^\infty \frac{1}{\sqrt{t}} \phi((x + \theta)/\sqrt{t}) \theta^{p-1}d\theta$$

$$= \frac{1}{\sqrt{t}} \phi(x/\sqrt{t}) - c^{p/2} \phi(x/\sqrt{t}) t^{(p-1)/2} \int_0^\infty e^{-ux/\sqrt{u^2/2}} u^{p-1}du$$

$$= \frac{1}{\sqrt{t}} \phi(x/\sqrt{t}) - \frac{c^{p/2} \Gamma(p)}{\sqrt{2\pi}} t^{(p-1)/2} e^{-x^2/4t} D_{-p}(x/\sqrt{t})$$

where $D_{-p}$ is the parabolic cylinder function. Let $b(t; c, p)$ be the unique solution to $h(x,t) = 0$ (Lerche (1986) shows the existence of such a unique solution). Then $b(t; c, p)$ satisfies

$$e^{-b^2(t;c,p)/4t} = \Gamma(p)(ct)^{p/2} D_{-p}(b(t;c,p)/\sqrt{t})$$

(5.1)

Some of the properties that solutions $b$ to such equations satisfy are:

1. $b(t; c, p)$ is infinitely often continuously differentiable

2. $b(t)/t$ is monotone increasing

3. $b(t)$ is convex

as shown in Lerche (1986). For our particular example there are a number of additional properties that $b(t; c, p)$ satisfies. Notice that for each $c > 0$ the boundary satisfies the scaling property $b(t; c, p) = b(ct; 1, p)/\sqrt{c}$. That is, if $b(t; c, p)$ is a solution to (5.1) then so is $b(ct; 1, p)/\sqrt{c}$ using the map $t \rightarrow ct$. Application of Lemma 6 below implies that for
each $p > 0$ the density function of the FPT to $b(t; c, p)$, $f(t; c, p)$, is in the scale family of distributions w.r.t. the parameter $c > 0$. Also, setting $g(t) = b(t; c, p)/\sqrt{t}$, we can rewrite (5.1) as

$$t = g^{-1}(u) = \frac{1}{c} \left[ \frac{1}{\Gamma(p)e^{u^2/4}D_{-p}(u)} \right]^{2/p}, \ u = g(t)$$

and differentiating w.r.t. $u$ we obtain

$$\frac{d}{du} g^{-1} = 2g^{-1}(u) \frac{D_{-p-1}(u)}{D_{-p}(u)} > 0, \ \forall u \in \mathbb{R}$$

The last expression being positive since $D_{-p} > 0$ for positive values of $p$. Therefore $g^{-1}(u)$ is monotone increasing which implies $b(t; c, p)/\sqrt{t}$ is a monotone increasing function. Furthermore, Corollary 2 below implies that for all $t > 0$ we have a ranking of the boundaries; $b(t; c_1, p) < b(t; c_2, p)$ for all $t, p > 0$ and $0 < c_1 < c_2$. Finally, using the asymptotic behavior of $D_{-p}(u)$, (A.10), for large values of $u$ we obtain the asymptotic behavior of $g^{-1}$ given by

$$g^{-1}(u) \sim \frac{1}{c} \left[ \frac{1}{\Gamma(p)e^{u^2/4}u^{-p}e^{-u^2/4}} \right]^{2/p} = \frac{u^2}{c\Gamma(p)^{2/p}}$$

Thus, the asymptotic behavior of $b$, for large $t$, is given by

$$b(t; c, p) \sim \sqrt{c\Gamma(p)^{1/p}}t$$
Next we examine the density $f(t; c, p)$. Using (1.24) and the properties (A.12) and (A.11) of the parabolic cylinder function, we have

\[
(1/2)h_x(b(t; c, p), t) = -\frac{g(t)}{2t} \phi(g(t)) - \frac{\Gamma(p)e^{p/2}(p-1)/2}{2\sqrt{2\pi}} \left[ -\frac{g(t)e^{-g(t)^2/4}}{2\sqrt{t}} D_{-p}(g(t)) + \frac{e^{-g(t)^2/4}}{\sqrt{t}} D'_{-p}(g(t)) \right]
\]

\[
= -\frac{g(t)}{2t} \phi(g(t)) + \frac{\Gamma(p)(ct)^{p/2}e^{-g(t)^2/4}}{2t\sqrt{2\pi}} D_{-p+1}(g(t))
\]

\[
= -\frac{\phi(g(t))}{2t} \left[ g(t) - \frac{D_{-p+1}(g(t))}{D_{-p}(g(t))} \right]
\]

\[
= \frac{p\phi(g(t))D_{-p-1}(g(t))}{2tD_{-p}(g(t))}
\]

\[
= \frac{1}{t} R(u)|_{u=b(t;c,p)/\sqrt{t}}
\]

where $R(u) = \frac{p\phi(u)D_{-p-1}(u)}{2D_{-p}(u)}$ and we have used (5.1) in the third equality above. Therefore

\[
tf(t; c, p) = R(b(t; c, p)/\sqrt{t}).
\]

For large $t$ we have

\[
f(t) \sim K(p)e^{-t(c/2)\Gamma(p)^2/t^3/2}
\]

where $K(p) = p/(2\Gamma(p)^{1/p}\sqrt{2c\pi})$, using the large time behavior of $b(t; c, p) \sim \sqrt{t}\Gamma(p)^{1/p}t$ and the asymptotic behavior of $D_{-p}$. To our knowledge this class of boundaries $b(t; c, p)$, $c, p > 0$ has not been explored thus far in the context of the FPT problem.

### 5.2 Polynomial Expansion of FPT density

The idea in this section is to expand the kernel of equation (2.37) in terms of a complete orthonormal basis in $L^2$ and show that $f$ belongs to the space spanned by this system. Then we use equation (2.37) to obtain a linear system for the coefficients of the expansion of $f$. More formally let $L^2(w) := \{ h : [0, \infty) \mapsto \mathbb{R}; \langle h, h \rangle_w < \infty \}$ be the $L^2$ inner product...
space equipped with the inner product $\langle x, y \rangle = \int_0^\infty x(z)y(z)w(z)dz$ and let $\phi_n^w$ be a complete orthonormal system for $L^2(w)$. Define

$$A^* := \{ \alpha \in \mathbb{R}; \|K(\alpha, t)\|_w < \infty \}$$

where $K$ is the kernel in equation (2.37). Thus, for $\alpha \in A^*$ we can write $\frac{K(\alpha, t)}{w(t)} = \sum_0^\infty a_n(\alpha)\phi_n^w(t)$ where the coefficients $a_n(\alpha)$ are given by

$$a_n(\alpha) = \int_0^\infty K(\alpha, t)\phi_n^w(t)dt$$

With $A$ defined as in (2.36) we introduce the following result:

**Lemma 5** Suppose $\alpha \in A \cap A^*$ with $b(t) \in C^{(1)}([0, \infty))$, $b(0) < 0$. Then

$$f(t) = \sum_0^\infty \langle f, L_n \rangle_w L_n(t)$$

$$\sum_0^\infty a_n(\alpha)\langle f, L_n \rangle_w = 1,$$

where $L_n$ are the Laguerre polynomials and $w(t) = e^{-t}$, provided that the series $\sum_0^\infty a_n(\alpha)$ converge absolutely.

**Proof.** Since $b(0) < 0$ we have $f(0) = 0$ and since $b \in C^{(1)}([0, \infty))$ $f$ is continuous and diminishes at infinity. It follows that $f$ is uniformly bounded and thus $\|f\|_w < \infty$. Therefore the coefficients $c_n := \langle f, L_n \rangle_w$ exist and $f(t) = \sum_0^\infty c_nL_n(t)$. Since $\alpha \in A \cap A^*$ it follows that $a_n(\alpha)$ exist for all $n$. Let $Z_M(\alpha) = \int_0^\infty \sum_M^\infty |a_n(\alpha)L_n(t)|e^{-t}f(t)dt$. Then, for large enough
Approximate Analytical Solutions to the FPT and IFPT Problems

We have

\[ Z_M(\alpha) = \sum_{M} \left| a_n(\alpha) \right| \int_{0}^{\infty} |L_n(t)| e^{-t} f(t) dt \]

\[ \leq C_1 \sum_{M} \left| a_n(\alpha)(1 + O(n^{-1/2})) \right| \int_{0}^{\infty} \frac{e^{t/2}}{t^{1/4}} e^{-t} dt \]

\[ \leq C_2 \sum_{M} \left| a_n(\alpha) \right| \leq \infty \]

where \( C_{1,2} \) are positive constants. The first inequality was obtained using the limiting behavior of the Laguerre polynomials for large \( n \)

\[ L_n(t) \sim \frac{e^{t/2}(1 + O(n^{-1/2})) \cos(2\sqrt{nt} - \frac{\pi}{4})}{\sqrt{\pi(nt)^{1/4}}} \]

(see Gradshteyn and Ryzhik (2000), 8.978), and the uniform bound of \( f \). The second inequality follows from the finiteness of the integral in the second inequality. From the first inequality above we can see that we can relax the assumption of absolute convergence of the series \( \sum_{M} a_n(\alpha) \) to absolute convergence of \( \sum_{M} a_n(\alpha)/n^{1/4} \). Next, rewrite equation (2.37) as

\[ 1 = \int_{0}^{\infty} e^{-\alpha b(t) - \frac{(\alpha^2 - 2t)^t}{2}} e^{-t} f(t) dt = \int_{0}^{\infty} \sum_{0}^{\infty} a_n(\alpha) L_n(t)e^{-t} f(t) dt = \]

\[ = \int_{0}^{\infty} \sum_{0}^{M} a_n(\alpha) L_n(t)e^{-t} f(t) dt + \int_{0}^{\infty} \sum_{M}^{\infty} a_n(\alpha) L_n(t)e^{-t} f(t) dt = \]

\[ = \sum_{0}^{M} a_n(\alpha) \int_{0}^{\infty} L_n(t)e^{-t} f(t) dt + \sum_{M}^{\infty} a_n(\alpha) \int_{0}^{\infty} L_n(t)e^{-t} f(t) dt = \]

\[ = \sum_{0}^{\infty} a_n(\alpha) c_n \]

where the third line follows from Fubini’s Theorem since \( Z_M(\alpha) < \infty \). This completes the proof. \( \square \)
Note that $A \cap A^*$ is a relatively large class. If $b(t) > c$, $\forall t \geq 0$, for some $c < 0$, then for all $\alpha \geq 1$ equation (2.37) holds since $b(t) + \alpha t > c + \alpha t > c$ and $(K(\alpha, t)e^t)^2 < e^{-2\alpha c}$ so that $K(\alpha, t)e^t$ can be expanded in terms of the Laguerre polynomials $L_n$. Thus in the case when $b \in C^1([0, \infty))$ and is uniformly bounded (below) then $\sum_0^\infty a_n(\alpha)c_n = 1$ for all $\alpha \geq 1$ provided that the coefficients $a_n(\alpha)$ form a uniformly convergent series.

**Example:** Let $b(t) = bt - a$, $a, b > 0$. For $\alpha > \sqrt{b^2 + 1} - b$ the conditions of Lemma 5 are satisfied and

$$a_n(\alpha) = \int_0^\infty e^{-(ab+\alpha^2/2)t}L_n(t)dt = \frac{2}{2\alpha b + \alpha^2 + 2} \left( 1 - \frac{2}{2\alpha b + \alpha^2 + 2} \right)^n$$

Since $0 < a_n(\alpha) < 1$ the coefficients are absolutely convergent and we have the equation:

$$\sum_0^\infty \left( 1 - \frac{2}{2\alpha b + \alpha^2 + 2} \right)^n c_n = e^{-\alpha a}(2\alpha b + \alpha^2 + 2)/2$$

In fact this equation can be solved exactly for $c_n$ by rewriting it as $\sum_0^\infty z^n c_n = h(z)$, expanding $h(z)$ in power series and matching the coefficients on both sides. Another way to look at this equation is to denote $\frac{\alpha^2 + 2ab + 2}{\alpha^2 + 2\alpha b} := z$ (note that $z > 1$) then the equation becomes

$$\sum_0^\infty z^{-n} c_n = e^{ab} \frac{z e^{-a\sqrt{b^2 z - b^2 +2}}}{z - 1}$$

The right hand side of this equation represents the unilateral $z$-transform.

We can also use the normalized Hermite polynomials $\tilde{H}_n(t) = \frac{H_n(t)}{2^{n/2}\sqrt{n!}}$ as a complete orthonormal basis in $L^2(e^{-t^2})$. In this case we need to extend the kernel $K(\alpha, t)$ and $f$ to be zero on $(-\infty, 0)$. Furthermore, this basis would restrict the class of functions for $b$. It is no longer sufficient that $b(t) + \alpha t$ be bounded. We would need $b(t) \sim t^2$ for large $t$ so that $K(\alpha, t)e^t \in L^2(e^{-t^2})$. In this case Lemma 5 would still hold (under the same assumptions as before) since $\tilde{H}_n(t) = O(e^{t^2/2})$ allows us to use Fubini’s theorem as in the proof above.
Thus, the Laguerre polynomials are a natural basis to use because they are orthogonal with respect to the density of the exponential distribution and the moment generating function of the Brownian motion is log-linear.

Another advantage of the Laguerre polynomials is given by the following result due to McCully (1960)

\[
T_n \left\{ \int_0^x g(t) \, dt \right\} = c_n - c_{n-1}, \quad n = 1, 2, \ldots
\]

with \( T_0 \{ \int_0^x g(t) \, dt \} = c_0 \), where \( T_n \) is the Laguerre transform \( T_n \{ g \} = \int_0^\infty e^{-x} L_n(x) g(x) \, dx \) and \( c_n \)'s are the coefficients of the Laguerre expansion of \( g \). This result holds for continuous (and more generally piecewise continuous) functions \( g \) satisfying \( g(x) = O(e^{ax}) \), \( a < 1 \) for large positive \( x \). Thus if \( f \), the density of \( \tau \), satisfies \( \int_0^\infty e^{-t} f^2(t) \, dt < \infty \), then

\[
f(t) = \sum_0^\infty c_n L_n(t)
\]

and the cumulative distribution function \( F \) satisfies

\[
F(t) = \sum_0^\infty k_n L_n(t), \quad k_0 = c_0, \quad k_n = c_n - c_{n-1}, \quad n = 1, 2, \ldots
\]

since \( F \in L^2(e^{-t}) \) (in fact \( \|F\|_w < 1 \)). It is sufficient that \( f(0) < \infty \) for \( f \) to be in \( L^2(e^{-t}) \). This is the case when \( b(0) < 0 \) and \( b(t) \) is monotone in the neighborhood of 0 as shown in Peskir (2002a).

Another quantity of interest is the Laplace transform of \( f \), \( \tilde{f}(z) := \int_0^\infty e^{-zt} f(t) \, dt \). Suppose \( z > 1/2 \), then we have

\[
\tilde{f}(z) = \int_0^\infty e^{-(z-1)t} e^{-t} f(t) \, dt = \int_0^\infty \sum_0^\infty \left( \frac{z-1}{z} \right)^n L_n(t) e^{-t} f(t) \, dt = \sum_0^\infty \left( \frac{z-1}{z} \right)^n c_n
\]

since \( e^{-(z-1)t} \in L^2(e^{-t}) \), \( z > 1/2 \) and \( T_n \{ e^{-at} \} = \frac{a}{a+1} \), \( a > -1 \), a result due to Erdelyi (1954). The last equality follows from Fubini’s theorem since \( \sum_0^\infty \left( \frac{z-1}{z} \right)^n \) are absolutely
convergent series for $z > 1/2$ and $L_n(t) = O(e^{t/2})$ for large $n$. Therefore, if we can obtain the coefficients $c_n$ in the Laguerre expansion of $f$ then we get for free the Laguerre expansion of the c.d.f. $F$ and the power series of the Laplace transform $\hat{f}(1/(1 - u)) = \sum_{n=0}^{\infty} u^n c_n$ for $u > -1$ where we have used the substitution $(z - 1)/z = u$.

We can apply the above technique to obtain approximation to the densities of future first passage times. Let $\tau_T := \inf\{t > T; W_t \leq b(t)\}$ together with the condition $X := W_T > b(T)$. Then we can write $\tau_T$ as $\tau_T = \inf\{u > 0; X + W_u \leq b(T + u)\}$ (where $W_u$ and $X$ are independent) and so conditional on $X = x > b(T)$ we have a version of equation (2.37) (assuming $b$ satisfies the usual boundedness conditions for some collection of $\alpha$’s).

$$\int_0^\infty e^{-\alpha b(T + s) - \frac{\alpha^2}{2} s} f(s; T|x) ds = e^{-\alpha x},$$

where $f(s|x)$ is the conditional density of $\tau_T$ given $X = x$. Integrating both sides w.r.t. the conditional density of $X|X > b(T)$ given by $\phi(x; T) = \frac{1}{\sqrt{2\pi T} \Phi(-b(T)/\sqrt{T})} e^{-\frac{x^2}{2\pi}}, x > b(T)$, where $\Phi$ is the standard normal c.d.f., we obtain the Fredholm equation:

$$\int_0^\infty e^{-\alpha b(T + s) - \frac{\alpha^2}{2} s} f(s; T) ds = \frac{1}{\sqrt{2\pi T} \Phi(-b(T)/\sqrt{T})} \int_0^\infty e^{-\alpha x} e^{-\frac{x^2}{2\pi}} dx = e^{\alpha^2 T/2} \frac{\Phi\left(-\frac{b(T) + \alpha T}{\sqrt{T}}\right)}{\Phi\left(-\frac{b(T)}{\sqrt{T}}\right)}$$

Thus, under the same regularity conditions of Theorem 2, we have a similar result: $f(t; T) = \sum_{0}^{\infty} c_n(T) L_n(t)$ where $c_n$ satisfy $\sum_{0}^{\infty} c_n(T) a_n(\alpha, T) = e^{\alpha^2 T/2} \frac{\Phi\left(-\frac{b(T) + \alpha T}{\sqrt{T}}\right)}{\Phi\left(-\frac{b(T)}{\sqrt{T}}\right)}$ and $a_n(\alpha, T)$ are the coefficients in the Laguerre expansion of $\exp\{-\alpha b(T + t) - t(\alpha^2/2 - 1)\}$.

We end the section with a numerical approach to approximate solutions to equation (5.3) using Tikhonov’s regularization technique (see Tikhonov (1963) or Groetsch (2007) and the references therein). We saw that under some conditions on the boundary $b(t)$ we can write the corresponding density $f$ as $f(t) = \sum_{0}^{\infty} c_n L_n(t)$ where $c_n$ satisfy the ill-posed problem $\sum_{0}^{\infty} a_n(\alpha) c_n = 1$ for all $\alpha \in \mathcal{A} \cap \mathcal{A}^*$. For large $N$ we can write $f(t) \approx \sum_{0}^{N} c_n L_n(t)$ and in
this section we look for optimal coefficients $c_1, \ldots, c_N$ such that $\sum_0^N a_n(\alpha)c_n \approx 1$. Because we have $N$ unknowns we pick $M \leq N$ values for $\alpha$, $\alpha_j$, $j = 1, \ldots, M$ in $A$ and we minimize $||Ac - 1||^2$ where the $M \times N$ matrix $A$ has entries $A_{jn} = a_n(\alpha_j)$ and the $N \times 1$ vector $c$ is given by $c = (c_1, \ldots, c_N)$. However, since we have infinitely many choices for $\alpha$, the matrix $A^T A$ may be ill-conditioned or nearly singular. To stabilize the solution we will use Tikhonov’s regularization technique by adding the regularization term $||Bc||^2$ and minimize $||Ac - 1||^2 + ||Bc||^2$, where $B$ is a suitably chosen matrix. We know this minimization problem has an explicit solution $c^*$ given by $c^* = (A^T A + B^T B)^{-1} B^T 1$. In the examples below we use the matrix $B = \sigma I$, where $I$ is the identity matrix. Here, $\sigma$ has the effect of smoothing the approximation to $f$. For values of $\sigma$ away from 0 the approximation is smoother while for values of $\sigma$ close to 0 the approximation is less smooth. For $\sigma = 0$ the minimization problem reduces to the original un-regularized problem which has the least squares solution provided that $(A^T A)^{-1}$ exists.

When the boundary is increasing fast e.g. $b(t) \sim t^p$, $p \geq 2$ then equation (2.37) holds for all real $\alpha$ and thus the corresponding density has to be of order $e^{-t^q}$, $q \geq p$ so that the integral in (2.37) exists for $\alpha < 0$. Therefore, for boundaries which increase fast, the corresponding densities are less smooth for large values of $t$ and we should use smaller values for $\sigma$ in the minimization problem (especially if we want a good approximation over a large time interval). Finally, in the examples below we use the following representation of the Laguerre polynomials 

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k$$

and thus the coefficients $a_n(\alpha)$ are given by 

$$a_n(\alpha) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} \int_0^\infty e^{-\alpha b(x) - x\alpha^2/2} x^k dx$$
The integral part in the above sum is computable explicitly for some boundaries. For example when \(b(t) = \sqrt{t} - a\) then

\[
a_n(\alpha) = e^{\alpha a} \sum_{m=0}^{n} (-1)^m \frac{(n)}{m!} \int_0^\infty e^{-\alpha \sqrt{t-t_0^2/2}} t^m dt
\]

\[
= \frac{2e^{\alpha a+1/4}}{\alpha^2} \sum_{m=0}^{n} \left( \frac{1}{m!} \right) \left( \frac{1}{\alpha^2} \right)^m \frac{(2m+1)!}{m!} D_{-2m-2}(1)
\]

Next we look at some examples. In these examples we are comparing the boundary obtained by solving equation (2.17) numerically using standard quadrature procedure and the boundary obtained by estimation of the Laguerre coefficients using Tikhonov’s regularization. The numerical procedure was implemented as follows. Let \(0 = t_0 < t_1 < \ldots < t_n = T\) be a partition of the interval \([0, T]\) such that \(t_{i+1} - t_i = d\), a constant. Since all of the boundaries considered below are monotone in the neighborhood of 0, are \(C^1\) functions, and are strictly negative at 0, we can set \(f(0) = 0\) (see Peskir (2002a)). We estimate \(f(t_i), i \geq 1\), recursively by discretizing the integral in (2.17). Thus, once we have obtained \(f(t_1), \ldots, f(t_i)\) we can compute \(f(t_{i+1})\) by

\[
\frac{df(t_{i+1})}{dt} = \Phi \left( \frac{b(t_{i+1})}{\sqrt{t_{i+1}}} \right) - \sum_{j=0}^{i} \Phi \left( \frac{b(t_{i+1}) - b(t_j)}{\sqrt{t_{i+1} - t_j}} \right) f(t_j)
\]

Note that the division by 2 on the left side of this equality comes from the limit

\[
\lim_{s \to t} \Phi \left( \frac{b(t) - b(s)}{\sqrt{t - s}} \right) = 1/2
\]

since \(b\) is differentiable.

**Example:** \(b(t) = t - 1\) with \(M = N = 100, \sigma = 0.001\) We have used 500 time points on the interval \([0, 1]\). The sum of squares between the densities is 0.1585 and the sum of squares for the cdf’s is 0.0009. Results are presented in Fig. 5.1.
Example: $b(t) = \log(t + 1) - 1$ with $M = N = 100$, $\sigma = 0.01$. We used 100 time points on the interval $[0, 1]$. The sum of squares between the densities is 0.1045 and the sum of squares for the cdf’s is 0.0025. Results are presented in Fig. 5.2.

Example: $b(t) = t^3 - t^2 - 1$ with $M = N = 100$, $\sigma = 0.001$. We used 100 time points on the interval $[0, 1]$. The sum of squares between the densities is 0.0381 and the sum of squares for the cdf’s is 0.0025. Results are presented in Fig. 5.2.
squares for the cdf’s is 0.0004. Results are presented in Fig. 5.3.

Figure 5.3: a) Numerical and Laguerre density and cdf; b) Difference of densities and cdf’s

**Example:** \( b(t) = \sqrt{t} - 1 \) with \( M = N = 100, \sigma = 0.01 \). We used 100 time points on the interval \([0, 1]\). The sum of squares between the densities is 0.6962 and the sum of squares for the cdf’s is 0.0159. Results are presented in Fig. 5.4.

Figure 5.4: a) Numerical and Laguerre density and cdf; b) Difference of densities and cdf’s

Note that in all of the above examples the approximation to the cdf is excellent even when
the corresponding density approximation is not. This is due to the smoothing property of the integral operator which takes the density \( f \) to the cdf \( F \).

### 5.3 Space and Time Change

Next we look at space/time changed Brownian motions and the reduction of the FPT and IFPT problems for these processes to the FPT and IFPT problems for the Brownian motion. Let \( B(t) \) be a positive, monotone increasing function with \( B(0) = 0 \) and \( g(x, t) \) be continuous, monotone increasing function in \( x \) with \( x \)-inverse \( g_x^{-1} \). Let \( X_t = g(W_B(t), t) \) where \( W \) is a standard Brownian motion. This class of processes includes GBM, O-U process and Brownian Bridge and more generally stochastic integrals with deterministic integrand and continuous Gauss-Markov processes (Doob (1949)). Suppose the first passage time

\[
\tau = \inf \left\{ t > 0; \ X_t \leq \hat{b}(t) \right\}
\]

has a distribution \( \hat{F}(t) = \mathbb{P}(\tau \leq t) \) Then

\[
\tau^* = B(\tau) = B \left( \inf \left\{ t; \ W_{B(t)} \leq g_x^{-1}(\hat{b}(t), t) \right\} \right) = \inf \left\{ B(t); \ W_B(t) \leq g_x^{-1}(\hat{b}(t), t) \right\} = \inf \left\{ u; \ W_u \leq g_x^{-1}(\hat{b}(B^{-1}(u)), B^{-1}(u)) \right\}
\]

has a distribution \( F(u) := \hat{F}(B^{-1}(u)) \). Thus the first and inverse passage problems for \( X_t \), with distribution \( \hat{F} \) and boundary \( \hat{b} \) is equivalent to the first and inverse passage problems for \( W_t \), with distribution \( F \) and boundary \( b(u) := g_x^{-1}(\hat{b}(B^{-1}(u)), B^{-1}(u)) \) and all the results for the Brownian motion problem pass, through a time change, to the first and inverse passage problems for the process \( X_t \).

Following this idea suppose the time change is actually monotone decreasing and let \( B(t) = 1/t \). Then, if \( \tau \) is the FPT of \( W \) to the boundary \( b \), we have

\[
\tau^* := 1/\tau = \sup \{ 1/t; W_t \leq b(t) \} = \sup \{ u > 0; aW_{1/u} \leq ub(1/u) \}
\]

\[
= \sup \{ u > 0; \hat{W}_u \leq ub(1/u) \}
\]
where the last equality is in distribution and $\hat{W}$ is a standard Brownian motion. The quantity $\tau^*$ represents the last exit time of $\hat{W}_u$ from the interval $(-\infty, ub(1/u)]$ and we can see that $\mathbb{P}(\tau^* < t) = \mathbb{P}(\tau > 1/t) = 1 - F(1/t)$. Thus the FPT and the last exit time are closely related quantities and studying the last exit time of a Brownian motion from a time dependent interval is equivalent to studying the FPT problem for the Brownian motion.

Let $F(u) =: F_\lambda(u)$ depend on some positive parameter $\lambda$ which comes either from $B$ or from $\hat{F}$ and suppose $F_\lambda(u)$ is in the scale family of distributions. Denote the corresponding boundary by $b_\lambda(u) := b(u)$. Then for $\beta > 0$ we have:

$$\beta \tau^* = \inf \{ \beta u > 0; W_u \leq b_\lambda(u) \} = \inf \{ t > 0; W_t \leq \sqrt{\beta} b_\lambda(t/\beta) \}$$

and since the distribution of $\beta \tau^*$ is given by $F_\lambda(u/\beta) = F_{\lambda/\beta}(u)$ with corresponding boundary $b_{\lambda/\beta}(u)$, from uniqueness of the boundaries we have $b_{\lambda/\beta}(u) = \sqrt{\beta} b_\lambda(t/\beta)$. Taking $\beta = \lambda$ we get $b_\lambda(t) = b_1(\lambda t)/\sqrt{\lambda}$. This result shows that in the scale family of distributions the inverse boundary problem is reduced to finding a single boundary, which we call base boundary. The result is stated for the base boundary corresponding to $\lambda = 1$ but we could use any boundary as the base boundary which results in the equality

$$b_\lambda(t) = \frac{b_{\lambda_0}(\lambda t)}{\sqrt{\lambda/\lambda_0}} \quad (5.4)$$

If we assume that $b_{\lambda}$ satisfies the scaling property (5.4) for all $\lambda, \lambda_0 > 0$, by reversing the above argument and using uniqueness of the distributions, it is easy to see that $F_{\lambda}$ is in the scale family of distributions. Thus we showed the following result:

**Lemma 6** For Brownian motion, $F_{\lambda}, \lambda > 0$ is in the scale family iff $b_{\lambda}(t) = b_1(\lambda t)/\sqrt{\lambda}$ for all $\lambda > 0$.

**Example:** Let $B(t) = \frac{1}{\lambda} \hat{F}(t), \lambda > 0$, then $F_{\lambda}(u) = \lambda u, u \leq \lambda$ is the c.d.f. of $U[0, \lambda]$
distribution, which is in the scale family of distributions. At the end of this section we show an approximation for the boundary of a uniform distribution on the unit interval.

Using the results in the previous section we can approximate $F(t) \approx \sum_{n=0}^{N} c_n L_n(t)$ using a Laguerre polynomials expansion and again, by reversing the time change, we have an approximation for $\hat{F}(t) \approx \sum_{n=0}^{N} c_n L_n(B(t))$. If $b_\lambda(t)$ satisfies the scaling property (5.4) then we target say $F_1$ from which we obtain $F_\lambda$ and $\hat{F}_\lambda$. Alternatively we can approximate $\hat{f}$ directly using the corresponding integral equation for $W_{B(t)}$:

$$\int_{0}^{\infty} e^{-ab(t)-B(t)\alpha^2/2} \hat{f}(t) dt = 1$$

which is obtained from equation (2.37) (or (2.38)) by the transformation $t \to B(t)$ and we follow the same steps as in the previous section to obtain an approximate polynomial expansion for $\hat{f}$. The only difference is in the choice of the complete orthogonal basis. If $B(t) \sim t^k$ then we can use Laguerre polynomials for $k \geq 1$ and Hermite polynomials for $k \geq 2$. The reason for direct approximation of $\hat{F}$ or $\hat{f}$ is that we may be interested in transforms of $\hat{f}$, such as the moment generating function, which cannot be obtained from the corresponding transforms of $f$ by reversal of the time change.

For the Brownian motion and the scale family of distributions we also have the following result.

**Corollary 2** Suppose the Brownian motion FPT distribution is in the scale family with parameter $\lambda$, corresponding boundary $b_\lambda$ and base boundary $b_{\lambda_0}$. Then

a) If $(t_0, b_{\lambda_0})$ is a point of interest on $b_{\lambda_0}$ the corresponding points of interest on $b_\lambda$, for all $\lambda > 0$, lie on the curve $(u, \alpha \sqrt{u})$ where $\alpha = \frac{b_{\lambda_0}}{\sqrt{t_0}}$.

b) On $[0, T]$ the function $\frac{b_{\lambda_0}(t)}{\sqrt{t}}$ is monotone increasing iff $b_{\lambda_1}(t) \leq b_{\lambda_2}(t)$ for all $0 < \lambda_1 \leq \lambda_2 \leq \lambda_0$

**Proof.** a) $b_\lambda(t_0 \frac{\lambda_0}{\lambda}) = \frac{b_{\lambda_0}(t_0)}{\sqrt{\lambda}} = \sqrt{\frac{\lambda_0}{\lambda}} b_0$ and the points $(t_0 \frac{\lambda_0}{\lambda}, \sqrt{\frac{\lambda_0}{\lambda}} b_0)$ give us the curve
(\(u, \alpha, \sqrt{u}\)) since \(0 < \lambda < \infty\)

b) Only if part:

\[
b_{\lambda_1}(t) = \frac{b_{\lambda_0}(\frac{\lambda_1 t}{\lambda_0})}{\sqrt{\frac{\lambda_1}{\lambda_0} t}} = \sqrt{t} \frac{b_{\lambda_0}(\frac{\lambda_1 t}{\lambda_0})}{\sqrt{\frac{\lambda_1}{\lambda_0} t}} \leq \sqrt{t} \frac{b_{\lambda_0}(\frac{\lambda_2 t}{\lambda_0})}{\sqrt{\frac{\lambda_2}{\lambda_0} t}} = b_{\lambda_2}(t)
\]

b) If part: Suppose \(\lambda_1 \leq \lambda_2\). Then

\[
\frac{b_{\lambda_0}(\frac{\lambda_1 t}{\lambda_0})}{\sqrt{\frac{\lambda_1}{\lambda_0} t}} = \frac{b_{\lambda_1}(t)}{\sqrt{t}} \leq \frac{b_{\lambda_2}(t)}{\sqrt{t}} = \frac{b_{\lambda_0}(\frac{\lambda_2 t}{\lambda_0})}{\sqrt{\frac{\lambda_2}{\lambda_0} t}} \tag{\[\square\]}
\]

Of course if \(\frac{b_{\lambda_0}(t)}{\sqrt{t}}\) is monotone increasing on \([0, T]\) then \(\frac{b_{\lambda}(t)}{\sqrt{t}}\) is increasing on \([0, T \frac{\lambda_0}{\lambda_2}]\) and \(\alpha.\sqrt{t}\) intercepts \(b_{\lambda}\) in exactly one point (excluding the zero) inside this interval if \(t_0 < T\). The first part of the Corollary says that if \(b_{\lambda_0}\) has a minimum at \((t_0, b_0)\) then all minimums lie on \(\alpha.\sqrt{t}\). In other words the intersection of \(b_{\lambda}(t)\) with \(\alpha.\sqrt{t}\) marks the minimum of \(b_{\lambda}\). The same applies for an inflection point. The above results allow us, for each distribution in the scale family, to target a single boundary and in doing so we can approximate explicitly the boundary for each parameter value of the underlying distribution. The idea is to use one of the Volterra integral equations to estimate the boundary with desired precision. We then fit an explicit approximation using a least squares procedure. Finally, using (5.4), we get an approximation for all parameter values. Note that if we approximate say \(b_1(t)\) by \(\hat{b}_1(t)\) on the interval \([0, T]\) then for any \(\lambda > 1\), \(\hat{b}_\lambda(t)\) may not be performing well for \(t > \frac{T}{\lambda}\). In general, as \(\lambda\) increases the approximation interval for \(b_{\lambda}\) decreases. However, we can chose the base boundary depending on the range of parameter values we are interested. From now on we assume the base boundary to be the boundary corresponding to \(\lambda = 1\). Furthermore, for any explicit approximation we have to use (1.30) which gives us the small time behavior of the boundary. This result could also be used for approximations of the boundary of the form \(-\sqrt{-2t \log F(t)} + h(t)\) with an appropriate function \(h\) which goes to 0 as \(t \downarrow 0\). Once
we obtain an approximation for $b_1(t)$ and thus for $b_\lambda(t)$ we can reverse the time (and space) change to obtain an approximation for $\hat{b}_\lambda(t)$. Next we look at two examples for the Brownian motion and distributions in the scale family.

**Inverse First Passage Time Numerical Examples**

As outlined in the previous section we will use equation (1.15) to obtain a numerical solution for $b(t)$ following the methodology of Chadam et al. (2006b) who propose a change of variable to deal with the singularity of the kernel in (1.15). We then fit an explicit approximation using least squares procedure. In the following notation we would not distinguish between the actual boundary $b(t)$ and its numerical approximation.

**Uniform distribution**

If $\tau$ is $U[0, 1]$ with boundary $b_1(t)$ then we know that $P(\tau \leq 1) = 1$. Thus $b_1(1) = \infty$ and from (1.30) we know $b_1(0) = 0$. We start with the following approximation which satisfies the conditions for the boundary at $t = 0$ and $t = 1$:

$$\tilde{b}_1(t) = a.\sqrt{t}\Phi^{-1}(bt^2 + (1 - b)t)$$

for some constants $a$ and $b$. The motivation for this functional form comes from equation (2.17). From this equation, since $\Phi < 1$, we have the inequality $\Phi(b_1(t)/\sqrt{t}) \leq F_1(t) = t$. Furthermore, since $\Phi(b_1(t)/\sqrt{t})$ is 0 at $t = 0$ and 1 at $t = 1$ we approximate this expression by a quadratic function $bt^2 + (1 - b)t$ which is less than $t$ on the interval $(0, 1)$. Finally, for more flexibility and a better fit we multiply by the parameter $a$. Note that any numerical solution to this boundary would converge at $t = 1$ thus we have excluded the last two percent of the points when estimating $a$ and $b$. The least squares procedure produced $a = 1.08634462$ and $b = 0.6048212$ with a sum of squared errors of 0.2967 for 20000 time points (excluding the last 400). The precision of the numerical solution is $10^{-10}$. Fig. 5.5 a) shows the numerical boundary and Fig. 5.5 b) shows the difference between the numerical and approximate boundaries as a function of time. Thus an approximation to the boundary
for $U[0, \lambda]$ distribution is given by

$$\tilde{b}_\lambda(t) = a\sqrt{t}\Phi^{-1}(b(\lambda t)^2 + \lambda(1-b)t)$$

Moreover, $b_1$ has a minimum point $(0.1766, -0.61574)$ and thus the minimums of $b_\lambda$ lie on the curve $-0.01465\sqrt{t}$. Fig. 5.5 a) compares $\tilde{b}_\lambda$ with $b_\lambda$ for several values of $\lambda$.

![Figure 5.5: a) U[0, 1] boundary; b) Difference of U[0, 1] boundary and $\tilde{b}_1$](image)

**Exponential distribution**

Suppose $\tau$ has Exp(1) distribution with corresponding boundary $b_1(t)$. Here we have used the small time behavior (1.30) of $b$ to define the following approximation

$$\tilde{b}_1(t) = \sqrt{-2t.\log F(t)} + a_1 t^{a_2} + a_3 t^{a_4}$$

where $F$ is the Exponential distribution cdf and the estimated constants are $a_1 = 0.5866, a_2 = 0.8341, a_3 = 0.1848, a_4 = 1.6387$, which were obtained by a fit to the numerical boundary on the interval $[0, 3]$ using 20000 points and precision $10^{-10}$. The sum of squared errors was $29.10^{-4}$. Fig. 5.6 a) shows the numerical boundary and Fig. 5.6 b) shows the fit of the ap-
proximation to the numerical boundary. The resulting approximation for \( b_\lambda(t) \) corresponding to \( \text{Exp}(\lambda) \) distribution is given by

\[ \tilde{b}_\lambda(t) = \frac{1}{\lambda} \sqrt{-2\lambda t \log F(\lambda t)} + a_1(\lambda t)^{a_2} + a_2(\lambda t)^{a_4} \]  

(5.5)

Moreover, since \( b_1 \) has a minimum point \((0.2457, -0.66506)\), the minimums of \( b_\lambda(t) \) lie on the curve \(-1.34171\sqrt{t}\) and occur at \( t = \frac{0.2457}{\sqrt{\lambda}} \). Fig. 5.7 b) compares \( \tilde{b}_\lambda \) with \( b_\lambda \) for several values of \( \lambda \).

Figure 5.6: a) \( \text{Exp}(1) \) boundary on [0,3]; b) Difference between \( \text{Exp}(1) \) boundary and \( \hat{b}_1 \)
Figure 5.7: a) Difference between $U(0, \lambda)$ and $\hat{b}_\lambda$; b) Difference between $Exp(\lambda)$ and $\hat{b}_\lambda$
Chapter 6

Conclusion

In the first part of this dissertation, we developed a new class of Volterra integral equations of the first kind for the distribution of the first passage time (FPT) of a standard Brownian motion to a regular boundary. This new class generalizes and unifies the class of all such previously known integral equations. Interestingly, this class arises through the optional stopping theorem applied to an interesting and new class of martingales generated by the parabolic cylinder functions. We demonstrated how this and more general classes of martingales can be constructed using the solutions to the heat equation on an infinite rod. Through the Abel integral transformation, we were able to prove uniqueness of a continuous solution to a subclass of integral equations in the case when the regular boundary is a well behaved function in the neighborhood of zero. Based on this uniqueness result, we were then able to consolidate the derivation of the FPT distribution to a set of transformed boundaries. These first-passage time distributions were expressed in terms of the original boundary and its FPT distribution function.

Furthermore, we generalized a class of Fredholm integral equations to the complex domain. These equations were then shown to provide a unified approach for computing the FPT distribution for linear, square root and quadratic boundaries. We believe that the
method can be more widely applied by searching for specific factorizations of the kernel that produce known transforms such as Mellin, Laplace, Hilbert and so on. Finally, for uniformly bounded continuous boundary functions, we demonstrated that there is a fundamental connection between the Volterra and the Fredholm integral equations studied in this work.

In the second part we examine a modification of the classical FPT problem, the randomized FPT or the matching distribution (MD) problem. Under this problem the object of interest is the random starting point, $X$, of the Brownian motion which is assumed to be independent of the Brownian filtration. This second source of randomness provides flexibility and allows us to take the boundary and the (unconditional) distribution of the FPT as inputs while seeking a matching distribution of random starting point. We obtained sufficient conditions for the existence and uniqueness of such a random variable $X$ and derived the Laplace and Hermite transforms of its density function (assuming it exists) using the new Volterra and Fredholm equations discussed in Chapter 2. These two transforms provide us with a semi-analytical solution to the MD problem and thus produce a partial solution to both the FPT and inverse FPT problems. Furthermore, we addressed the relationship between different boundaries and their corresponding matching distributions. Finally, we motivated the use of the solution to the MD problem to attack the classical FPT problem by deriving integral equations in the FPT setting involving new quantities and using the randomization technique to obtain known transforms of these quantities. In the case of the linear boundary we obtained analytical results for the matching distribution under a large class of unconditional distributions (which is an infinite mixture of gamma distributions). Finally, we derived a connection between the FPT with a random slope and the FPT with a random intercept.

Furthermore, we applied the randomized FPT of Chapter 3 to model the mortality of a Swedish cohort using a linearly drifted Brownian motion with a random intercept to represent
the 'health' process of an individual. In this setting the FPT is the time of death and its distribution was approximated by fitting a mixture of gamma distributions to the mortality data. We investigated the trade-off between the level of precision in the fit and the maximum amount of volatility allowed. Moreover, we motivated the use of this dynamical model and its structural setting to price mortality linked financial products. For this purpose, in the model, the 'risk-neutral' measure is induced by a slope change while keeping the distribution of the random intercept unchanged. Thus, the transition to a new measure has a natural interpretation and by changing the slope we simply express our believe that individuals die faster/slower under the 'risk-neutral' measure.

In the last part we analyzed the expansion of the FPT density with respect to the Laguerre polynomials through the Fredholm equation of Chapter 2, which is particularly well suited for these orthogonal polynomials because of the exponential form of its kernel. We derived a linear system for the coefficients in the expansion of the FPT density and employed the regularization method of Tikhonov (1963) to deal with the ill-posedness of the problem. The number of examples presented compare the numerical results based on the Laguerre polynomials expansion and the numerical solution to one of the Volterra integral equations for a number of boundaries.

We ended the work with an investigation of space/time changed Brownian motions for which the FPT and inverse FPT problems can be reduced to those for a standard Brownian motion. For the IFPT problem we demonstrated that if the distribution of the FPT is in the scale family then the corresponding boundary satisfies the same scaling property. Thus, in the case of the scale family of distributions the IFPT problem is reducible to finding a single, base boundary. This idea was applied to the exponential $Exp(\lambda)$ and uniform $U([0, \lambda])$ distributions by finding the base boundary (for $\lambda = 1$) using a numerical solution to one of the Volterra integral equations, fitting a functional form for the base boundary and obtaining the functional form for the boundary corresponding to a general $\lambda$. 
There are several directions remaining open for future research.

- The first is clear but difficult: how can this larger (uncountably infinite), new class of Volterra integral equations be used to extract the FPT distribution? One way is to explore the flexibility of the parabolic cylinder function and its connection to other special functions. Furthermore, the continuum of Volterra equations provides more flexibility for manipulation such as integration and differentiation w.r.t. the parameter $p$.

- The search for new Volterra equations of the first kind is related to identifying analytical solution to the heat equation. The search for such solutions, which generate kernel functions with known properties, is another topic for future research. Any linear combination of solutions to the heat equation is also a solution and possibly, in the limit, one can obtain Volterra equations with more informative kernels.

- We saw that taking the limit $y \uparrow b(t)$ in (2.7), with $p = 1$, produced the Volterra equation of the second kind (2.14). This motivates the investigation of this limit for the equations with $p > 1$. We suspect that in the computation of this limit for $p > 1$, we can obtain new Volterra equations of the second kind and such equations are known to exhibit unique solutions and are generally easier to deal with than the Volterra equations of the first kind. However, such equations would hold for a restricted class of boundary functions.

- The class of Volterra equations is also a useful tool for the inverse first passage time problem. Though, in this context, the equations are highly non-linear the generalization of this class provides flexibility for their manipulation which could extract new information.

- In the MD framework, the derivation of the matching distribution for non-linear boundaries is an open problem. Although we have derived the Laplace transform of the
matching density it is not clear how to invert it since it is in an integral form. Perhaps a different approach is needed to address the non-linear boundary case.

- Randomization of the slope of a linear boundary is another topic for future research. In this case, using the FPT distribution (with a constant slope) and using the equality (2.36) we have

\[
\frac{f(t|\alpha)}{f_c(t)} = e^{\alpha c - \frac{\alpha^2 t}{2}}
\]

where \(\alpha\) is the slope and \(c < 0\). If we assume that there exists a density function \(g(\alpha)\) such that \(\int_0^\infty f(t|\alpha)g(\alpha)d\alpha = f(t)\), where \(f\) is the target unconditional distribution of the FPT, we obtain

\[
\frac{f(t)}{f_c(t)} = \int_0^\infty e^{\alpha c - \frac{\alpha^2 t}{2}} g(\alpha)d\alpha
\]

By integrating out \(c\) or \(t\) we could obtain the Laplace transform of \(g\).

- In the randomized FPT for a linear boundary with unit slope we have the equality \(W_\tau = \tau - X\). If we can obtain the distribution of \(X\) for arbitrary target distribution (of \(\tau\), \(f\), then we can calculate the distribution of \(W_\tau\). In this case, the stopping time, \(\tau\), could be a solution to Skorohod’s problem which, for a given probability measure \(\mu\), seeks a stopping time \(\tau_{\mu}\) such that \(W_{\tau_{\mu}} \sim \mu\). First, this motivates the investigation of the existence of a random variable \(X\). We showed in Chapter 3 that, in the case of a linear boundary, existence of \(X\) is equivalent to the complete monotonicity property of the the function

\[
r(\alpha) := \int_0^\infty e^{-\alpha t - \alpha^2 t/2} f(t)dt
\]

In order to verify the complete monotonicity of \(r\) we can differentiate under the integral for a large class of densities \(f\). Furthermore, recognizing the exponential term under the integral as the generating function of the Hermite polynomials, it follows that our \(n\)-th derivative would be an integral of a quantity involving the Hermite polynomial of
degree $n$ and the density $f$ which is computable analytically in the case when $f$ is a gamma/erlang density function. Second, if we could show that the distribution of $W_\tau$ can be replicated by some distribution for $\tau$ and the corresponding distribution for $X$ then we would have a potential solution to Skorohod’s problem.
Appendix A

Supplementary Results

Lemma 7 Suppose $b : (0, T] \to \mathbb{R}$ is an increasing continuous function on $(0, \epsilon]$ for some $0 < \epsilon < 1$ with $b(0) = -\infty$. Let $h : \mathbb{R}^+ \to \mathbb{R}$ and $h(x) = O(e^{ax^2})$ for large $x > 0$ and some $0 < a < 1/2$. Define the first passage time $\tau := \{s > 0; W_s \leq b(s)\}$. Then

$$\int_0^\epsilon |h(-b(s))|F(ds) < \infty \quad (A.1)$$

where $F$ is the distribution of $\tau$.

Proof. Without loss of generality we can assume $-b(t) \gg 0$ for $t \leq \epsilon$. Define the the first-passage time $\tau_b(s) := \{t > 0; W_t \leq b(s)\}$ for $s < \epsilon$. Since $b(t) < b(s)$ for $t < s < \epsilon$ then $F(s) < F_{\tau_b(s)}(s) = 2\Psi(-b(s)/\sqrt{s})$ for all $s \leq \epsilon$. Let $g(s) = h(-b(s))$, $s < \epsilon$ and fix $s_1 < \epsilon$ and $\delta > 0$ be such that $k_{a,\delta}(s_1) := e^{ab^2(s_1)} - \delta > 0$. Define $s_n$ such that $e^{ab^2(s_n)} = k_{a,\delta}(s_1) + n\delta$. Since $e^{ab^2(s)}$ is monotone decreasing on $(0, \epsilon)$ with $e^{ab^2(0)} = \infty$ then $s_n \downarrow 0$ is a monotone decreasing sequence. Let

$$g_\delta(s) = \delta \sum_{n=2}^\infty 1(s \leq s_{n-1}) + (k_{a,\delta}(s_1) + \delta)1(s \leq \epsilon)$$

Then $0 < |g(s)| \leq Me^{ab^2(s)} \leq Mg_\delta(s)$, $s \in (0, \epsilon]$, for some $M > 0$, and by the dominated
convergence theorem and the definition of \( g \), there exists an \( \epsilon > 0 \) such that

\[
\int_0^\epsilon |g(s)|F(ds) \leq M \int_0^\epsilon g\delta F(ds) = M\delta \sum_{n=2}^\infty F(s_{n-1}) + Me^{ab^2(s_1)} F(\epsilon)
\]

\[
\leq 2M\delta \sum_{n=1}^\infty \Psi(b(s_n)/\sqrt{s_n}) + C
\]

\[
\leq \frac{2\sqrt{2M\delta}}{\sqrt{\pi}} \sum_{n=1}^\infty \phi(b(s_n)/\sqrt{s_n}) + C
\]

\[
\leq \frac{2M\delta}{\pi} \sum_{n=1}^\infty e^{-b^2(s_n)/2} + C
\]

\[
= \frac{2M\delta}{\pi} \sum_{n=1}^\infty (k_{a,\delta}(s_1) + n\delta)^{-1/(2a)} + C
\]

\[
< \infty
\]

where \( C = Me^{ab^2(s_1)} F(\epsilon) \). The third line holds since \( \Psi(x) \leq \phi(x), \ x > 0 \) while the last inequality follows from \( a < 1/2 \). This completes the proof. \( \square \)

In particular, for \( y \in \mathbb{R}, \ k > 0 \), we have

\[
\int_0^\epsilon (y - b(s))^k F(ds) < \infty \quad (A.2)
\]

**Lemma 8** Let \( b(t) \) be a continuously differentiable function on \((0,T] \) with \(-\infty < b(0) < 0\) satisfying \( \lim_{t\to 0} |b'(t)|t^\epsilon < \infty \) for some \( 0 < \epsilon < 1/2 \). Then, for all \( 0 \leq s \leq t, \):

\[
\left| \frac{b(t) - b(s)}{\sqrt{t-s}} \right| < C
\]

and

\[
\left| \frac{b(t) - b(s)}{t-s} \right| t^\epsilon < K
\]

for some positive constants \( C \) and \( K \)

**Proof.** Since \( b \) is continuous on \([0,T] \) the results hold for all \( 0 \leq s < t \leq T \). Since \( b \) is
differentiable on \((0, T]\) the results hold on the curve \(T \geq s = t > 0\). We only need to check the case \(s = t = 0\). For \(s = 0\) and \(t \downarrow 0\) we have that

\[
\lim_{t \downarrow 0} \frac{|b(t) - b(0)|}{\sqrt{t}} = \lim_{t \downarrow 0} \frac{|b(t) - b(0)|}{t} = \lim_{t \downarrow 0} |b'(t)|t^{1/2-\epsilon} = 0
\]

and

\[
\lim_{t \downarrow 0} \frac{|b(t) - b(0)|}{t} = \lim_{t \downarrow 0} |b'(t)|t^{\epsilon} < \infty \quad \square
\]

**Lemma 9** Suppose \(v, d, B, C > 0\) are positive constants and \(A \in \mathbb{R}\) and such that \(d + s + A + B\sqrt{C + s} > 0\) for any \(s > 0\). Then the Laplace transform of

\[
h(t) := \frac{B}{2\Gamma(v)\sqrt{\pi}} \int_0^t (t-x)^{-3/2} x^v e^{-C(t-x)-x(d+A)} - \frac{x^2B^2}{8(t-x)} dx
\]

is given by

\[
\tilde{h}(s) = (d + A + s + B\sqrt{s + C})^{-v}
\]

for \(s > 0\).

**Proof.** We compute the Laplace transform directly:

\[
\tilde{h}(s) = \int_0^{\infty} e^{-st}h(t)dt = \frac{B}{2\Gamma(v)\sqrt{\pi}} \int_0^{\infty} x^v e^{-x(d+A+s)} \int_0^{\infty} u^{-3/2} e^{-u(C+s) - x(d+A) - \frac{x^2B^2}{8(t-x)}} du dx
\]

\[
= 2\frac{B}{2\Gamma(v)\sqrt{\pi}} \int_0^{\infty} x^v e^{-x(d+A+s)} \left( \frac{x^2B^2}{4(C+s)} \right)^{-1/4} K_{-1/2} \left( 2 \left( \frac{x^2B^2(C+s)}{4} \right)^{1/2} \right) dx
\]

\[
= \frac{B}{2\Gamma(v)\sqrt{\pi}} \int_0^{\infty} x^{v-1} e^{-x(d+A+s+B\sqrt{s+C})} dx
\]

\[
= (d + A + s + B\sqrt{s + C})^{-v} \quad \square
\]

where we have used equation (3.471(9)) from Gradshteyn and Ryzhik (2000) in the second line above.
A.0.1 Parabolic Cylinder Function

Differential equations leading to parabolic cylinder functions:

\[
\frac{d^2 u}{dz^2} + \left( p + \frac{1}{2} - \frac{z^2}{4} \right) u = 0
\]  
(A.3)

The solutions are \( u = D_p(z), \ D_p(-z), \ D_{-p-1}(iz), \ D_{-p-1}(-iz). \)

Integral representation for \( p < 0: \)

\[
D_p(z) = \frac{e^{-z^2/4}}{\Gamma(-p)} \int_0^\infty e^{-xz - x^2/2} x^{-p-1} dx
\]  
(A.4)

Connection with other functions:

\[
D_n(z) = 2^{-n/2}e^{-z^2/4}H_n \left( \frac{z}{\sqrt{2}} \right)
\]  
(A.5)

\[
D_p(z) = 2^{1/4+p/2}W_{1/4+p/2,-1/4} \left( \frac{z^2}{2} \right) z^{-1/2}
\]  
(A.6)

\[
D_{-1/2}(z) = \sqrt{z\pi/2}K_{1/4} \left( \frac{z^2}{4} \right)
\]  
(A.7)

\[
D_{-2}(z) = e^{z^2/4} \left( e^{-z^2/2} - \sqrt{2\pi}z \Phi(-z) \right)
\]  
(A.8)

where \( H_n \) is the Hermite polynomial of degree \( n \), \( W \) is the Whittaker function, \( K \) is the modified Bessel function of the third kind.

Assymptotic expansions:

\[
D_p(z) \sim \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{izp}z^{-p-1}e^{z^2/4}, \ |z| \uparrow \infty, \ \pi/4 < |\arg(z)| < 5\pi/4 \]  
(A.9)

\[
D_p(z) \sim z^p e^{-z^2/4}, \ |z| \uparrow \infty, \ |\arg(z)| < 3\pi/4 \]  
(A.10)
A Supplementary Results

Other properties:

\[ D_{p+1}(z) - zD_p(z) + pD_{p-1}(z) = 0 \]  \hspace{1cm} (A.11)

\[ \frac{d}{dz}D_p(z) = \frac{1}{2}zD_p(z) - D_{p+1}(z) \]  \hspace{1cm} (A.12)

\[ \frac{d}{dz}e^{-z^2/4}D_p(z) = e^{-z^2/4}D_{p+1}(z) \]  \hspace{1cm} (A.13)

\[ \int_0^\infty e^{-z^2/4}D_{-p}(z)dz = \sqrt{\pi}2^{-p/2-1/2} = D_{-(p+1)}(0) \]  \hspace{1cm} (A.14)


\textbf{A.0.2 Airy function}

Airy function on the complex plain has the integral representation:

\[ Ai(x) = \frac{1}{2\pi i} \int_C e^{t^3/3 - xt}dt \]  \hspace{1cm} (A.15)

where the integral is over a path \( C \) with end points \( \infty e^{-\pi i/3} \) and \( \infty e^{\pi i/3} \).
Bibliography


