Jumping in line

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The Black-Scholes model (1973) has played a pivotal role in the development of option pricing theory. Much effort has been directed at extending this basic model to achieve closer agreement with market prices. However, no single model has won similar widespread recognition. The proliferation of pricing models is due not only to the search for numerical efficiency but also to the fundamental issue that has to be addressed when discriminating between different models. If two models agree on all prices of European-style claims at all maturities, they are not necessarily equivalent – they may forecast qualitatively different dynamics of option prices (Carr & Madan, 2000), produce different hedge ratios, show discrepancies in the prices of path-dependent options and disagree on what is the optimal exercise policy for American-style options.

Derman & Kani (1994, 1998) and Derman, Kani & Chriss (1996) have proposed explaining the observed deviations by postulating that the stock price process is a diffusion with continuous paths while the volatility depends on the stock price and calendar time. An alternative is to use stochastic volatility models that are in good agreement with historical time series, such as the Garch model studied by Duan (1996). But these models fail to capture the steepening tendency of the volatility smiles for short-dated options, which is better explained by jump processes for the stock price process. A jump model with an elegant economic justification is based on the variance-gamma (VG) process, introduced by Madan, Carr & Chang (1998, see also Carr *et al*, 2000). Conceivably, a second generation pricing framework will eventually emerge that combines the salient features of all these approaches.

In this article, we focus on the VG jump model and describe a simple numerical technique that can be regarded as a replacement of the binomial lattice approximations to accommodate jumps. The closed-form solution for European-style options derived in Meyer & Van der Hoek (1997) allows one to calibrate the VG model easily. However, the pricing problem for exotic path-dependent options has not been solved in analytic closed form. Methods such as Monte Carlo simulations or the solution of integro-differential equations can be used but their implementation presents challenges. The method we propose exploits a fundamental connection between the approximation scheme known as the "method of lines" and jump processes, from where we derive the name we propose, "model of lines". The method of lines was first introduced in the financial literature by Carr (1998) as an approximation methodology for pricing American-style options. Within the method of lines, time derivatives are replaced by finite differences, while derivatives with respect to stock price are kept intact. Consequently, the pricing equation is reduced to an inhomogeneous ordinary differential-difference equation. In most cases, the resulting equation can be solved analytically. Otherwise, numerical integration methods can be used. We demonstrate that a modified version of the method of lines, which incorporates a scaling of the stock price from one time step to the next, yields exact prices – up to rounding errors – to European-style options and extends to barrier, American-style and Bermudan options. Our model of lines yields exact solutions to the VG model for path-dependent options with payout contingent on information on the lines only, such as discretely monitored barrier and Bermudan options and special values of one of the VG parameters. American-style options are priced only approximately as the exercise boundary is piecewise constant in this scheme. The case of a general VG model is within reach of Richardson extrapolation methods.

In the following, we review the financial interpretation of the method of lines in terms of randomised maturity options as developed by Carr (1998), introduce the proposed model of lines that is appropriate for VG models, discuss barrier and Bermudan options, and elaborate on extrapolation methods. We refer the reader to Albanese, Jaimungal & Rubisov (2000) for a more thorough discussion of implementation details.

Jumps and the method of lines

The method of lines is an approximation scheme for solving quite general partial differential equations (PDEs). In contrast to lattice pricing models, only the time variable is discretised, while stock prices are continuous. This partial discretisation leads to a chain of ordinary differential equations (ODEs) along the lines to be solved backwards from maturity. The method of lines has been found to be an efficient numerical scheme for solving free boundary value problems, such as the pricing of American-style option contracts (Meyer & Van der Hoek, 1997). A semi-explicit solution to the sequence of ODEs arising from the method of lines version of the Black-Scholes PDE was given in Carr & Faguet (1994). Carr (1998) gives an intriguing financial interpretation of the method of lines that established a connection with random maturity contracts and has motivated us to extend the method to jump processes.

The price $P^{(n)}(S, K)$ of a European-style option maturing in $n\Delta t$ units of time is given by:

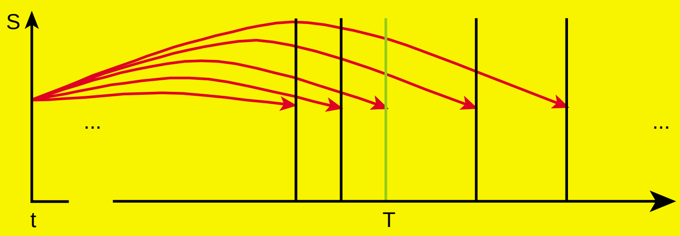
$$\left\{\frac{\sigma^2}{2}S^2 d_{SS} + r\,S d_S - r\right\}P^{(n)}(S,K) = \frac{1}{\Delta t}\left(P^{(n)}(S,K) - P^{(n-1)}(S,K)\right) \tag{1}$$

subject to the appropriate terminal and boundary conditions. For example, the conditions for a put option are as follows:

$$P^{(0)}(S,K) = (K-S)_+,\quad \lim_{S\to\infty} P^{(n)}(S,K) = 0,\quad \lim_{S\to 0} P^{(n)}(S,K) = e^{-r\,n\Delta t}K \tag{2}$$

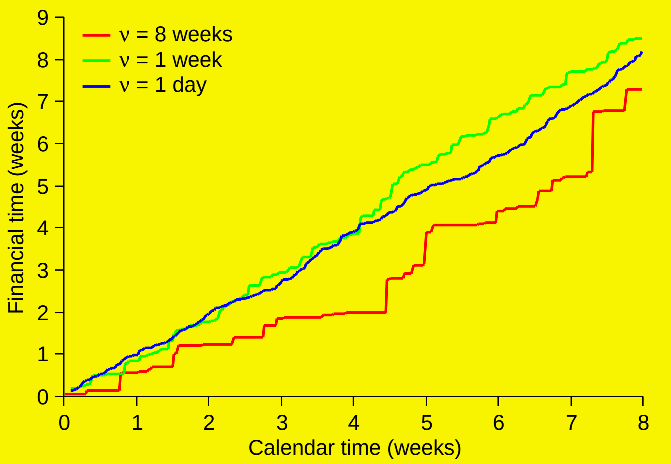
As usual $(a)_+$ is equal to a if $a > 0$ and zero otherwise. Carr (1998) notices that the resulting prices can be interpreted as prices of claims of ran-

1. Standard method of lines



The standard method of lines yields exact prices for an option that matures at a random time that is distributed according to an Erlang distribution

2. Gamma process sample paths that describe financial time



The graph shows several gamma process sample paths that describe financial time. Notice that these paths are different from typical diffusion process paths. Also, as ν tends towards zero, the path becomes more deterministic

dom maturity τ (see figure 1) following the Erlang distributed with density function:

$$\mathcal{P}\left(\tau \in (g, g+dg)\right) = \frac{g^{n-1}e^{-g/\Delta t}}{(n-1)!\,\Delta t^{n}}\,dg \tag{3}$$

In the limit $\Delta t \to 0$ the above density reduces to a delta function. The expiry of the option is then certain to occur at time $T = n\Delta t$ and the price converges to the Black-Scholes limit. Notice that (3) can be viewed as the density for a gamma process evaluated at time $n\Delta t$ with variance rate $\nu = \Delta t$ and mean rate 1.

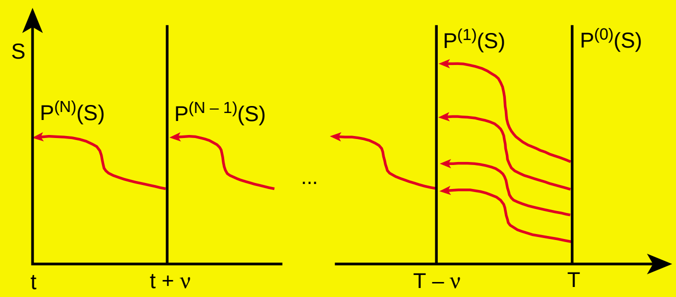
Instead of considering random maturity claims, we can equivalently consider claims of fixed maturity, whereby the underlying asset is subject to stochastic time changes. Stochastic time changes are related to jump processes, and since the Erlang distribution is a particular case of the gamma distribution, the underlying jump process is the VG process. This defines what we call the model of lines.

To make the connection with the VG model more precise, one has to ensure that stock prices drift at the risk-neutral rate. The stock price, in the risk-neutral measure, following a VG process, is given by:

$$S_t = S_{t_0}\exp\left\{\omega(t-t_0) + X\left(\Gamma(t-t_0,\nu);\theta,\sigma\right)\right\} \tag{4}$$

Here, $X(\tau;\theta,\sigma)$ denotes a Brownian process evaluated at time τ with drift θ and volatility σ; $\Gamma(\tau,\nu)$ is a gamma process evaluated at time τ with variance rate ν (and mean rate 1); and ω is chosen so that the discounted stock

3. Pictorical representation of the model of lines



The prices on the current line are obtained by solving an ordinary differential equation with source given by the discounted price on the previous line evaluated at a scaled spot price

price is a martingale (see, for instance, Madan, Carr & Chang, 1998), ie, risk-neutrality is satisfied:

$$\mathbb{E}\left[S_t\right] = e^{(t-t_0)r}S_{t_0} \Rightarrow$$
$$\omega = r + \frac{1}{\nu}\ln\left(1-\left(\theta+\frac{1}{2}\sigma^2\right)\nu\right) \tag{5}$$

Notice that, conditional on the gamma time, the stock price follows a lognormal process. Consequently, each unit of calendar time supports $\Gamma(t,\nu)$ units of "financial time" (see figure 2). Furthermore, in financial time, $\ln(S_t/S_0)$ is a Brownian process with drift θ and volatility σ^2. As a result, the price of a European-style contingent claim, maturing at time Δt with payout $h(S_{t+\Delta t})$, conditional on financial time is:

$$p(S_t,g) = \mathbb{E}\left[h(S_{t+\Delta t})\middle|g\right]$$
$$= e^{\left(\theta+\frac{1}{2}\sigma^2\right)g}P_{BS}\left(e^{\omega\Delta t}S_t, g, \left(\theta+\frac{1}{2}\sigma^2\right), \sigma\right) \tag{6}$$

where $P_{BS}(S, g, r, \sigma)$ denotes the Black-Scholes price of the plain vanilla option maturing at time g. It is straightforward to show that the conditional price satisfies the following variation of the Black-Scholes equation:

$$\left(-\partial_g + D_S\right)p(S_t,g) = 0 \tag{7}$$

with boundary condition:

$$\lim_{g\to 0} p(S_t,g) = h\left(e^{\omega\Delta t}S_t\right) \tag{8}$$

and the operator D_S is defined as follows:

$$D_S \equiv \frac{1}{2}\sigma^2 S^2 d_{SS} + \left(\theta+\frac{1}{2}\sigma^2\right)S d_S \tag{9}$$

The unconditioned price of the European-style claim is obtained by applying the gamma density:

$$f_{\Gamma(\Delta t,\nu)}(g) = \frac{g^{\frac{\Delta t}{\nu}-1}e^{-g/\nu}}{\Gamma\left(\frac{\Delta t}{\nu}\right)\nu^{\frac{\Delta t}{\nu}}} \tag{10}$$

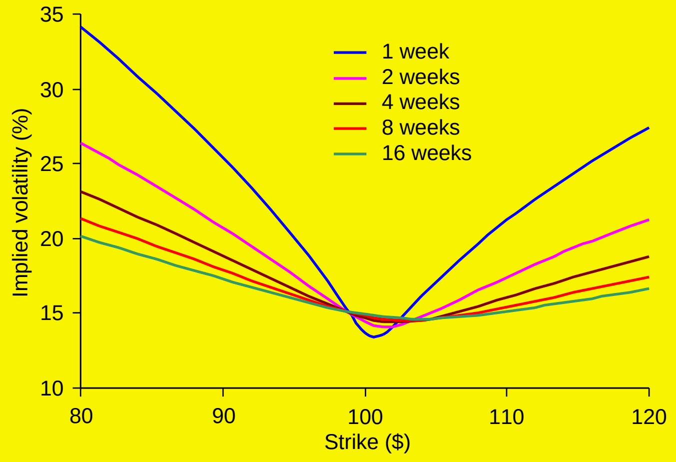
to the conditional price, performing the integration over all financial times and discounting the result:

$$P(S_t) = e^{-r\Delta t}\int_0^\infty dg\, f_{\Gamma(\Delta t,\nu)}(g)\,p(S_t,g) \tag{11}$$

Now consider the special case $\nu = \Delta t$. In this case, the Γ distribution reduces to the Poisson exponential distribution, and the unconditioned price reduces to:

$$P(S_t) = e^{-r\Delta t}\int_0^\infty dg\,\frac{e^{-g/\Delta t}}{\Delta t}\,p(S_t,g) \tag{12}$$

4. The implied smiles of the variance-gamma model



The implied smiles of the variance-gamma model for maturities of one, two, four, eight and 16 weeks with parameters: $\nu = \Delta t = 1$ week, $\sigma = 15\%$, $\theta = -20\%$ and $r = 5\%$

Applying the integral kernel appearing in the right-hand side of (12) to the equation in (7), one finds that:

$$-\frac{1}{\Delta t}\left[P(S_t) - \lim_{g \to 0} p(S_t, g)\right] + D_S P(S_t) = 0 \qquad (13)$$

The limit can be calculated by using the boundary condition in (8) to obtain:

$$-\frac{1}{\Delta t}\left[P(S_t) - e^{-r\Delta t} h\left(e^{\omega \Delta t} S_t\right)\right] + D_S P(S_t) = 0 \qquad (14)$$

This procedure can then be applied recursively to obtain the price of any payout that matures in multiples of Δt, and is given succinctly by the following differential-difference equations:

$$D_S P^{(n)}(S) = \frac{1}{\Delta t}\left(P^{(n)}(S) - e^{-r\Delta t} P^{(n-1)}\left(e^{\omega \Delta t} S\right)\right) \qquad (15)$$

$$P^{(0)}(S) = h(S) \qquad (16)$$

This system of ODEs is shown in figure 3.

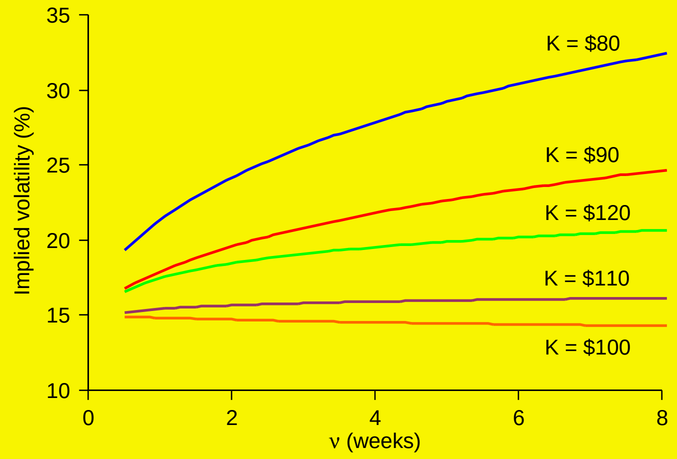
Within the model of lines, information between time steps is unavailable. Consequently, any option that is contingent on that information, such as an American-style option, cannot be priced exactly using this methodology. However, Bermudan options, which can be exercised at times $n\Delta t$, can be priced exactly as described below.

Let $\tilde{P}(S_t)$ denote the price of the plain-vanilla option given by equation (14). The price of the exotic, $P(S_t)$, at the current time step, with rebate $R(S_t)$ and lower exercise level $S_{E.B.}$, will then be:

$$P(S_t) = \begin{cases} \tilde{P}(S_t), S_t > S_{E.B.} \\ R(S_t), S_t \le S_{E.B.} \end{cases} \qquad (17)$$

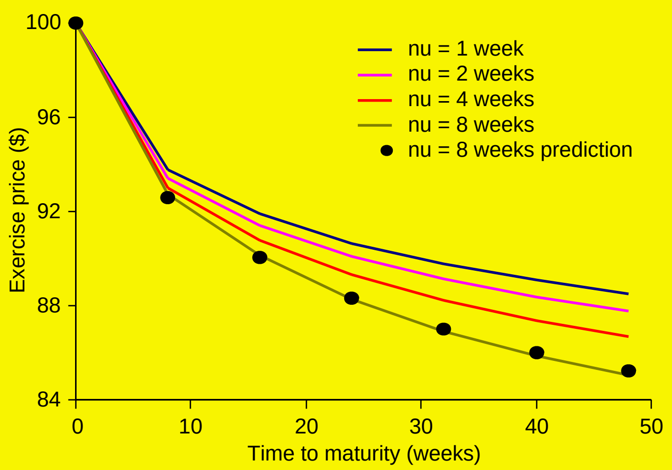
A down-and-out barrier is obtained by taking $S_{E.B.} = H$ as the lower barrier, and the rebate $R(S_t) = R$. A Bermudan put can be priced by choosing $S_{E.B.}$ such that $\tilde{P}(S_{E.B.}) = K - S_{E.B.}$ and rebate $R(S_t) = (K - S_t)_+$, and enforcing a smooth pasting condition. The interested reader is referred to Albanese, Jaimungal & Rubisov (2000) for the details. Generalisations to double barriers and other exotics are fairly straightforward. According to the standard dynamic programming procedure, once the price of the exotic is obtained on the current line, that price is treated as the payout to obtain the exotic price on the previous line and so on. The algorithm allows for efficient implementations, whereby errors are independent of the spacing between lines and are only due to the ODE integration.

5. Extrapolation of the implied volatility of one-month European options for a variance-gamma model



Extrapolation of the implied volatility of one-month European options for a variance-gamma model with $\nu = 8$ weeks using a fit to volatilities obtained with $\nu = \Delta t = 1$, 2 and 4 weeks. The model parameters were $\sigma = 15\%$, $\theta = -20\%$ and $r = 5\%$ and the spot was taken to be \$100

6. Boundary of a Bermudan option that can be exercised every eight weeks



The boundary of a Bermudan option that can be exercised every eight weeks plotted for several values of $\nu = \Delta t$. The black dots show the boundary with $\nu = 8$ weeks obtained by extrapolation using the first three boundaries. The model parameters were $\sigma = 15\%$, $\theta = -20\%$ and $r = 5\%$ and the spot was taken to be \$100

There are several important features of the model of lines that should be elaborated on. First, since trading occurs in calendar time, not in financial gamma time, the underlying stock in financial time does not need to satisfy a martingale condition resulting from arbitrage considerations. Hence, it is not surprising that the drift θ that appears in (9) is not the risk-free rate r. Second, in the operator D_S, there is no constant term, ie, the term $-rP$ in the usual Black-Scholes equation is missing. On reflection, it is clear that such a term must be absent because the discounting occurs in real time and not financial time. Third, once the price on one line is known, the price on the previous line is determined from an option with a scaled spot and discounted price. The discounting of the price is natural, and can be thought of as the spot price of the next line. The scaling of the spot itself can be understood from the fact that although the drift of the stock in financial time is not equal to the risk-free rate, risk neutrality must still be enforced. Consequently, across each line additional drifting must be imposed.

Volatility smiles for a VG process with $\sigma = 15\%$, $\theta = -20\%$ and $\Delta t = 1$ week are plotted in figure 4. We have calculated the relative error between the implied volatilities obtained using the exact prices in Madan, Carr & Chang (1998) and those obtained using the model of lines. The largest relative error for the smiles in figure 4 was found to be ~ $10^{-3}\%$ while the average relative error over the smiles was found to be ~ $10^{-5}\%$. There was little difference in computation time between the two pricing schemes.

Extrapolation techniques

Although the time step dictates the ν parameter in the VG model, it is possible to use the model of lines to obtain approximate prices to VG models in which ν is different from Δt. Just as Carr (1998) demonstrated that Richardson extrapolation to $\nu = 0$ reproduced the Black-Scholes value when applying the method of lines, we propose to use an extrapolation scheme to obtain the prices of options for $\nu \neq \Delta t$. In figure 5, we have plotted the exact implied volatilities for one-month European-style options with various strike levels as a function of ν. Quadratic polynomials in $\ln \nu$ were used to fit the first three points at $\nu = \Delta t = 1$, 2 and 4 weeks, and extrapolate to the fourth point at $\nu = 8$ weeks. The fitting curves also plotted in figure 5 demonstrate a reliable prediction. The absolute error in predicting the implied volatility at-the-money was found to be negligible. The worst case turned out to be that of the out-of-the-money put struck at 80% of the spot, for which the absolute error in implied volatility is 0.09%. Our conclusion is that extrapolation allows for the pricing of any VG model using the model of lines, but the prices thus obtained are approximate.

Just as in the case of plain-vanilla options, it is possible to use extrapolation to obtain the prices when $\nu \neq \Delta t$ in the exotic case. In figure 6, the boundary of a Bermudan option that can be exercised every eight weeks is plotted as a function of $\nu = \Delta t = 1, 2, 4$ and 8 weeks. The black dots in figure 6 form the predicted eight-week boundary obtained by extrapolation from the first three points. The extrapolation is based on a fit to a quadratic polynomial of $\ln \nu$ to the first three boundaries. The errors obtained by this extrapolation method are minimal, with a maximum absolute error of \$0.18 for the longest maturity option. The at-the-money prices fitted to a linear function of ν are displayed in figure 7. The errors are once again negligible, with the longest maturity option being underpriced by \$0.02.

Conclusion

We propose a generalisation of the method of lines, which produces exact prices for the VG process. The method admits a suggestive financial interpretation, applies to a large, although not exhaustive, family of VG models and is able to reproduce a large variety of implied volatility shapes. For this class of VG models, we are able to price exactly in terms of the solutions to ODEs, and any payout is contingent only on price levels at the lines, including Bermudan and barrier options. General VG models are within reach of extrapolation methods. The numerical method is efficient and simple to implement. ■

Claudio Albanese is a professor of mathematics, Sebastian Jaimungal is an NSERC postdoctoral fellow and Dmitri Rubisov is a research associate at the University of Toronto. The authors also work for Math Point, a consulting and software start-up company in Toronto. Claudio Albanese and Sebastian Jaimungal would like to thank the Natural Science and Engineering Research Council of Canada for financial support. We thank Peter Carr and the referee for helpful comments. All remaining mistakes are our own
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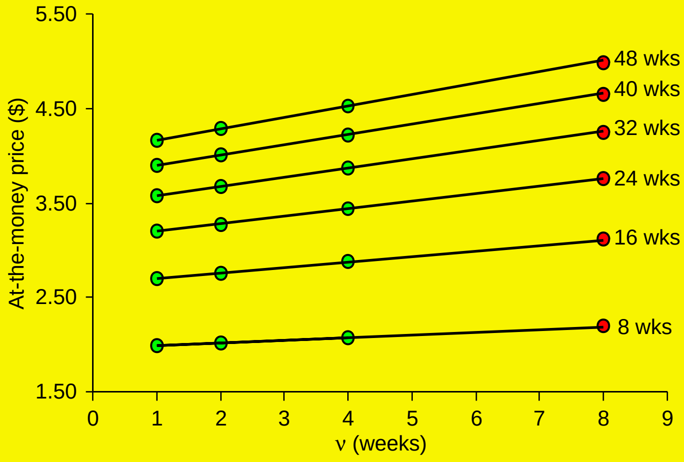
7. Prices of at-the-money options



The prices of the at-the-money options whose boundaries are shown in figure 6. The lines indicate a fit to the first three prices extrapolated to the fourth

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