Tuesday, June 01, 2010
11:32 AM

Value of options in Heston.

$$\frac{dS_t}{S_t} = r\,dt + \sqrt{v_t}\,dW_t \longrightarrow dx_t = (r - \tfrac{1}{2}v_t)\,dt + \sqrt{v_t}\,dW_t$$

$$dv_t = \kappa(\theta - v_t)\,dt + \eta\sqrt{v_t}\,dB_t$$

$$d[W,B]_t = \rho\,dt$$

$$g_t = e^{-r(T-t)}\,\mathbb{E}_t^{\mathbb{Q}}[\,\varphi(S_T)\,] \quad ,$$

$\underline{r=0}$:
$$h_t = \mathbb{E}_t^{\mathbb{Q}}[\,\varphi(S_T)]$$

$$h_t = h(t, \overset{\ln S_t}{x_t}, v_t)$$

h_t is a $\mathbb{Q}$-mtg $\therefore \mathbb{E}_t[dh_t] = 0$

$$dh_t = (\partial_t + \mathcal{L})h\,dt + \sqrt{v_t}\,\partial_x h\,dW_t + \eta\sqrt{v_t}\,\partial_v h\,dB_t$$

$$\Rightarrow (\partial_t + \mathcal{L})h = 0 \quad ; \quad h(T,x,v) = \varphi(e^x) = \int_{-\infty}^{\infty} e^{i\omega x}\,\hat{\varphi}(\omega)\,\frac{d\omega}{2\pi}$$

where
$$\mathcal{L} = -\tfrac{1}{2}v\,\partial_x + \tfrac{1}{2}v\,\partial_{xx} + \kappa(\theta - v)\partial_v + \tfrac{1}{2}\eta^2 v\,\partial_{vv} + \rho\eta v\,\partial_{xv}$$

value instead $h(T,x,v) = e^{i\omega x}$

assume affine $h(t,x,v) = e^{A_t + B_t x + C_t v}$

$$0 = (\partial_t + \mathcal{L})h = \underbrace{(\cdot)}_{0} + \underbrace{(\cdot)}_{0}\,x + \underbrace{(\cdot)}_{0}\,v$$

$$(\dot{A} + \dot{B}x + \dot{C}v)$$

$$-\tfrac{1}{2}v\,B + \tfrac{1}{2}v\,B^2 + \kappa(\theta - v)\,C + \tfrac{1}{2}\eta^2 v\,C^2 + \rho\eta v\,BC = 0$$

$\dot{B} = 0 \Rightarrow B = \text{const} = i\omega$

$$\dot{C} - \tfrac{1}{2}B + \tfrac{1}{2}B^2 - \kappa C + \tfrac{1}{2}\eta^2 C^2 + \rho\eta B C = 0$$

$$\Rightarrow \dot{C} + \tfrac{1}{2}\eta^2 C^2 + C(\rho\eta i\omega - \kappa) - \tfrac{1}{2}(i\omega + \omega^2) = 0$$

$$\dot{C} = -\tfrac{1}{2}\eta^2 \underbrace{(C - a_+)(C - a_-)}_{\text{roots are fn of } \omega.}$$

$$\dot{C}\,\frac{1}{(C-a_+)(C-a_-)} = -\tfrac{1}{2}\eta^2$$

$$\dot{C}\left(\frac{1}{C-a_+} - \frac{1}{C-a_-}\right)\frac{1}{a_+-a_-} = -\tfrac{1}{2}\eta^2$$

$$\ln\left(\frac{\overset{0}{C_T} - a_+}{C_t - a_+}\right) - \ln\left(\frac{\overset{0}{C_T} - a_-}{C_t - a_-}\right) = -\tfrac{1}{2}\eta^2 (a_+ - a_-)(T-t)$$

$$C_t = \frac{(\ ) - (\ )\, e^{-\frac{1}{2}\eta^2 (a_+ - a_-)(T-t)}}{(\ ) - (\ )\, e^{-\frac{1}{2}\eta^2 (a_+ - a_-)(T-t)}}$$

$$\dot{A} + \kappa\theta C = 0 \Rightarrow \underset{\hookrightarrow 0}{A_T} - A_t + \kappa\theta \int_t^T C_u\, du = 0$$

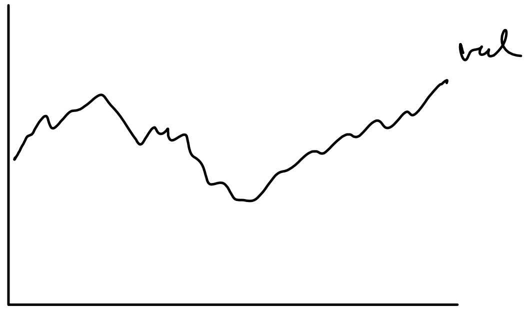
$\longrightarrow$ get affine Fns!!!

the price of $e^{i\omega x}$ is $e^{A_t + B_t x + C_t v}$

the price $\Phi(e^x) = \phi(x)$ is $\displaystyle\int_{-\infty}^{\infty} e^{A_t + B_t x + C_t v} \cdot \hat{\phi}(\omega)\,\frac{d\omega}{2\pi}$

$$\phi(x) = \int_{-\infty}^{\infty} e^{i\omega x}\,\hat{\phi}(\omega)\,\frac{d\omega}{2\pi}$$

How do jumps in asset price affect this method?



$$\frac{dS_t}{S_t} = r\,dt + \sqrt{v_t}\,dW_t$$

$$dW_t \to \rho B_t + \sqrt{1-\rho^2}\,B_t^{\perp}$$

$$dv_t = \kappa(\theta - v_t)\,dt + \eta\sqrt{v_t}\,dB_t$$

$$d[B,W]_t = \rho\,dt$$

$$\mathbb{E}[\varphi(S_T)]$$

$$= \mathbb{E}[\,\mathbb{E}[\varphi(S_T) \mid \mathcal{F}_0 \vee \sigma((B_s)_{0\le s\le T})]\,]$$

$$S_T = S_0 \exp\left\{\int_0^T (r - \tfrac{1}{2}v_u)\,du + \rho\int_0^T \sqrt{v_u}\,dB_u + \sqrt{1-\rho^2}\int_0^T \sqrt{v_u}\,dB_u^{\perp}\right\}$$

$$= S_0 \exp\left\{\int_0^T (r - \tfrac{1}{2}(1-\rho^2)v_u)\,du + \int_0^T \sqrt{(1-\rho^2)v_u}\,dB_u^{\perp} - \tfrac{1}{2}\int_0^T \rho^2 v_u\,du + \int_0^T \sqrt{\rho^2 v_u}\,dB_u\right\}$$

cond on $\sigma((B_s)_{0\le s\le T})$,

$$S_T \overset{d}{=} \bar{S}_0\, e^{(r - \frac{1}{2}\bar{\sigma}^2)(T\ \ ) + \bar{\sigma}\sqrt{T}\, Z}$$

$$Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1), \quad \bar{\sigma}^2 = \frac{(1-\rho^2)}{T}\int_0^T v_u\,du$$

$$\bar{S}_0 = S_0\, e^{-\frac{1}{2}\int_0^T \rho^2 v_u\,du + \int_0^T \sqrt{\rho^2 v_u}\,dB_u}$$

$$\mathbb{E}[\varphi(S_T)] = \mathbb{E}[\,P_{BS}(\bar{S}_0, \bar{\sigma})\,]$$ mixing method.

what would jumps do?