

Efficient Bayesian inference for stochastic volatility models with ensemble MCMC methods

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Abstract

In this paper, we introduce efficient ensemble Markov Chain Monte Carlo (MCMC) sampling methods for Bayesian computations in the univariate stochastic volatility model. We compare the performance of our ensemble MCMC methods with an improved version of a recent sampler of Kastner and Fruwirth-Schnatter (2014). We show that ensemble samplers are more efficient than this state of the art sampler by a factor of about 3.1, on a data set simulated from the stochastic volatility model. This performance gain is achieved without the ensemble MCMC sampler relying on the assumption that the latent process is linear and Gaussian, unlike the sampler of Kastner and Fruwirth-Schnatter.

The stochastic volatility model is a widely-used example of a state space model with non-linear or non-Gaussian transition or observation distributions. It models observed log-returns $y = (y_1, \dots, y_N)$ of a financial time series with time-varying volatility, as follows:

$$Y_i|x_i \sim N(0, \exp(c + \sigma x_i)), \quad i = 1, \dots, N \quad (1)$$

$$X_1 \sim N(0, 1/(1 - \phi^2)) \quad (2)$$

$$X_i|x_{i-1} \sim N(\phi x_{i-1}, 1) \quad (3)$$

Here, the latent process x_i determines the unobserved log-volatility of y_i . Because the relation of the observations to the latent state is not linear and Gaussian, this model cannot be directly handled by efficient methods based on the Kalman filter.

In a Bayesian approach to this problem, we estimate the unknown parameters $\theta = (c, \phi, \sigma)$ by sampling from their marginal posterior distribution $p(\theta|y)$. This distribution cannot be written down in closed form. We can, however, write down the joint posterior of θ and the log-volatilities $x = (x_1, \dots, x_N)$, $p(\theta, x|y)$ and draw samples of (θ, x) from it. Discarding the x coordinates in each draw will give us a sample from the marginal posterior distribution of θ .

To sample from the posterior distribution of the stochastic volatility model, we develop two new MCMC samplers within the framework of ensemble MCMC, introduced by Neal (2010). The key idea underlying ensemble MCMC is to simultaneously look at a collection of points (an “ensemble”) in the space we are sampling from, with the K ensemble elements chosen in such a way that the density of interest can be simultaneously evaluated at all of the ensemble elements in less time than it would take to evaluate the density at all K points separately.

Previously, Shestopaloff and Neal (2013) developed an ensemble MCMC sampler for non-linear, non-Gaussian state space models, with ensembles over latent state sequences, using the embedded HMM (Hidden Markov Model) technique of Neal (2003), Neal et al. (2004). This ensemble MCMC sampler was used for Bayesian inference in a population dynamics model and shown to be more efficient than methods which only look at a single sequence at a time. In this paper we consider ensemble MCMC samplers that look not only at ensembles over latent state sequences as in Shestopaloff and Neal (2013) but also over a subset of the parameters. We see how well both of these methods work for the widely-used stochastic volatility model.

1 Bayesian inference for the stochastic volatility model

Bayesian inference for the stochastic volatility model has been extensively studied. In this paper, we focus on comparisons with the method of Kastner and Fruwirth-Schnatter (2014). This state-of-the-art method combines the method of Kim et al. (1998) with the ASIS (Ancillary Sufficiency Interweaving Strategy) technique of Yu and Meng (2011). Kastner and Fruwirth-Schnatter’s method consists of two parts. The first is an update of the latent variables x and the second is a joint update of θ and the latent variables x . We improve this method here by saving and re-using sufficient statistics to do multiple parameter updates at little additional computational cost.

1.1 A linear Gaussian approximation for sampling latent sequences

The non-linear relationship between the latent and the observation process prohibits the direct use of Kalman filters for sampling the latent variables x_i . Kim et al. (1998) introduced an approximation to the stochastic volatility model that allows using Kalman filters to draw samples of x_i which can be later reweighed to give samples from their exact posterior distribution. This approximation proceeds as follows. First, the observation process for the stochastic volatility model is written in the form

$$\log(y_i^2) = c + \sigma x_i + \zeta_i \tag{4}$$

where ζ_i has a $\log(\chi_1^2)$ distribution.

Next, the distribution of ζ_i is approximated by a ten-component mixture of Gaussians with mixture weights π_k , means m_k and variances $\tau_k^2, k = 1, \dots, 10$. The values of these mixture weights, means and variances can be found in Omori (2007). At each step of the sampler, at each time i , a single component of the mixture is chosen to approximate the distribution of ζ_i by

drawing a mixture component indicator $r_i \in \{1, \dots, 10\}$ with probabilities proportional to

$$P(r_i = k | y_i, x_i, c, \sigma) \propto \pi_k \frac{1}{\tau_k} \exp((\log(y_i^2) - c - \sigma x_i)^2 / \tau_k^2) \quad (5)$$

Conditional on r_i , the observation process is now linear and Gaussian, as is the latent process:

$$\log(Y_i^2) | x_i, r_i, c, \sigma \sim N(m_{r_i} + c + \sigma x_i, \tau_{r_i}^2), \quad i = 1, \dots, N \quad (6)$$

$$X_1 | \phi \sim N(0, 1/(1 - \phi^2)) \quad (7)$$

$$X_i | x_{i-1} \sim N(\phi x_{i-1}, 1) \quad (8)$$

Kalman filtering followed by a backward sampling pass can now be used to sample a latent sequence x . For a description of this sampling procedure, see Petris et al. (2009).

The mis-specification of the model due to the approximation can be corrected using importance weights

$$w^{(l)} = \frac{\prod_{i=1}^N f(y_i | x_i^{(l)}, c^{(l)}, \sigma^{(l)})}{\prod_{i=1}^N (\sum_{k=1}^{10} \pi_k g(\log(y_i^2) | x_i^{(l)}, c^{(l)}, \sigma^{(l)}, m_k, \tau_k))} \quad (9)$$

where f is the $N(0, \exp(c + \sigma x_i))$ density, g is the $N(m_k + c + \sigma x_i, \tau_k)$ density and the index l refers to a draw. Posterior expectations of functions of θ can then be computed as $\sum w^{(l)} f(\theta^{(l)})$, with $\theta^{(l)}$ the draws.

We note that, when doing any other updates affecting x in combination with this approximate scheme, we need to continue to use the same mixture of Gaussians approximation to the observation process. If an update drawing from an approximate distribution is combined with an update drawing from an exact distribution, neither update will draw samples from their target distribution, since neither update has a chance to reach equilibrium before the other update disturbs things. We would then be unable to compute the correct importance weights to estimate posterior expectations of functions of θ .

1.2 ASIS updates

ASIS methods (Yu and Meng (2011)) are based on the idea of interweaving two parametrizations. For the stochastic volatility model, these are the so-called non-centered (NC) and centered (C) parametrizations. The NC parametrization is the one in which the stochastic volatility model was originally presented above. The C parametrization for the stochastic volatility model is

$$Y_i | \tilde{x}_1 \sim N(0, \exp(\tilde{x}_i)), \quad i = 1, \dots, N \quad (10)$$

$$\tilde{X}_1 \sim N(c, \sigma^2 / (1 - \phi^2)) \quad (11)$$

$$\tilde{X}_i | \tilde{x}_{i-1} \sim N(c + \phi(\tilde{x}_{i-1} - c), \sigma^2) \quad (12)$$

The mixture of Gaussians approximation for C is the same as for NC.

Kastner and Fruwirth-Schnatter (2014) propose two new sampling schemes, GIS-C and GIS-NC, in which they interweave these two parametrizations, using either the NC or C parameterization as the baseline. The authors report a negligible performance difference between using NC or C as the baseline. For the purposes of our comparisons, we use the method with NC as the baseline, GIS-NC, which proceeds as follows.

1. Draw x given θ, r, y using the linear Gaussian approximation update (NC)
2. Draw θ given x, r, y using a Metropolis update (NC)
3. Move to C by setting $\tilde{x} = c + \sigma x$
4. Draw θ given \tilde{x}, r, y using a Metropolis update (C)
5. Move back to NC by setting $x = \frac{\tilde{x}-c}{\sigma}$
6. Redraw the mixture component indicators r given θ, x, y .

Theorem 4 of Yu and Meng (2011) establishes a link between ASIS and the PX-DA (Parameter Expansion-Data Augmentation) method of Liu and Wu (1999). In the case of the stochastic volatility model, this means that we can view the ASIS scheme for updating x and θ as a combination of two updates, both done in the NC parametrization. The first of these draws new values for θ conditional on x . The second draws new values for both x and θ , such that when we propose to update c to c^* and σ to σ^* , we also propose to update the sequence x to $x^* = ((c + \sigma x) - c^*)/\sigma^*$. For this second update, the Metropolis acceptance probability needs to be multiplied by a Jacobian factor $(\sigma/\sigma^*)^N$ to account for scaling σ . A joint translation update for c and x has been previously considered by Liu and Sabatti (2000) and successfully applied to the stochastic volatility model. Scale updates are considered by Liu and Sabatti (2000) as well, though they do not apply them to the stochastic volatility model.

The view of ASIS updates as joint updates to θ and x makes it easier to see why ASIS updates improve efficiency. At first glance, they look like they only update the parameters, but they actually end up proposing to change both θ and x in a way that preserves dependence between them. This means that moves proposed in ASIS updates are more likely to end up in a region of high posterior density, and so be accepted.

Kastner and Fruwirth-Schnatter (2014) do a single Metropolis update of the parameters for every update of the latent sequence. However, we note that given values for the mixture indices r , y and x , low-dimensional sufficient statistics exist for all parameters in the centered parametrization. In the non-centered parametrization, given r , y and x , low-dimensional sufficient statistics exist for ϕ . We propose doing multiple Metropolis updates given saved values of these sufficient statistics (for all parameters in the case of C and for ϕ in the case of NC). This allows us to reach equilibrium given a fixed latent sequence at little computational cost since additional updates have small cost, not dependent on N . Also, this eliminates the need to construct complex proposal schemes, since with these repeated samples the algorithm becomes less sensitive to the particular choice of proposal density.

The sufficient statistics in the case of NC are

$$t_1 = \sum_{i=1}^N x_i^2, \quad t_2 = \sum_{i=2}^N x_{i-1}x_i, \quad t_3 = x_1^2 + x_N^2 \quad (13)$$

with the log-likelihood of ϕ as a function of the sufficient statistics being

$$\log(L(\phi|t)) = (1/2)\log(1 - \phi^2) - (1/2)(\phi^2(t_1 - t_3) - 2\phi t_2 + t_1)$$

In the case of C the sufficient statistics are

$$\tilde{t}_1 = \sum_{i=1}^N \tilde{x}_i^2, \quad \tilde{t}_2 = \sum_{i=2}^{N-1} \tilde{x}_i^2, \quad \tilde{t}_3 = \sum_{i=2}^N \tilde{x}_{i-1}\tilde{x}_i, \quad \tilde{t}_4 = \sum_{i=2}^{N-1} \tilde{x}_i, \quad \tilde{t}_5 = \tilde{x}_1 + \tilde{x}_N \quad (14)$$

with the log-likelihood as a function of the sufficient statistics being

$$\begin{aligned} \log(L(c, \phi, \sigma^2|\tilde{t})) &= -(N/2)\log(\sigma^2) + (1/2)\log(1 - \phi^2) \\ &\quad - (1/2)(\tilde{t}_1 + \phi^2\tilde{t}_2 - 2\phi\tilde{t}_3 - 2c\phi^2\tilde{t}_4 - 2c(\tilde{t}_4 + \tilde{t}_5) \\ &\quad + 4c\phi\tilde{t}_4 + 2c\phi\tilde{t}_5 + (N - 1)(c(\phi - 1))^2 + c^2(1 - \phi^2))/\sigma^2 \end{aligned} \quad (15)$$

The details of the derivations are given in the Appendix.

2 Ensemble MCMC methods for stochastic volatility models

The general framework underlying ensemble MCMC methods was introduced by Neal (2010). An ensemble MCMC method using embedded HMMs for parameter inference in non-linear, non-Gaussian state space models was introduced by Shestopaloff and Neal (2013). We briefly review ensemble methods for non-linear, non-Gaussian state space models here.

Ensemble MCMC builds on the idea of MCMC using a temporary mapping. Suppose we are interested in sampling from a distribution with density $\pi(z)$ on \mathcal{Z} . We can do this by constructing a Markov chain with transition kernel $T(z'|z)$ with invariant distribution π . The temporary mapping strategy takes T to be a composition of three stochastic mappings. The first mapping, \hat{T} , takes z to an element w of some other space \mathcal{W} . The second, \bar{T} , updates w to w' . The last, \check{T} , takes w' back to some $z' \in \mathcal{Z}$. The idea behind this strategy is that doing updates in an intermediate space \mathcal{W} may allow us to make larger changes to z , as opposed to doing updates directly in \mathcal{Z} .

In the ensemble method, the space \mathcal{W} is taken to be the K -fold Cartesian product of \mathcal{Z} . First, z mapped to an ensemble $w = (z^{(1)}, \dots, z^{(K)})$, with the current value z assigned to $z^{(k)}$, with $k \in \{1, \dots, K\}$ chosen uniformly at random. The remaining elements $z^{(j)}$ for $j \neq k$ are chosen

from their conditional distribution under an ensemble base measure ζ , given that $z^{(k)} = z$. The marginal density of an ensemble element $z^{(k)}$ in the ensemble base measure ζ is denoted by $\zeta(z^{(k)})$. Next, w is updated to w' using any update that leaves invariant the *ensemble density*

$$\rho(w) = \rho((z^{(1)}, \dots, z^{(K)})) = \zeta((z^{(1)}, \dots, z^{(K)})) \frac{1}{K} \sum_{i=1}^K \frac{\pi(z^{(k)})}{\zeta_k(z^{(k)})} \quad (16)$$

Finally, a new value z' is chosen by selecting an element $z^{(k)}$ from the ensemble with probabilities proportional to $\pi(z^{(k)})/\zeta_k(z^{(k)})$. The benefit of doing, say Metropolis, updates in the space of ensembles is that a proposed move is more likely to be accepted, since for the ensemble density to be large it is enough that the proposed ensemble contains at least some elements with high density under π .

In Shestopaloff and Neal (2013), we consider an ensemble over latent state sequences x . Specifically, the current state, (x, θ) , consisting of the latent states x and the parameters θ is mapped to an ensemble $y = ((x^{(1)}, \theta), \dots, (x^{(K)}, \theta))$ where the ensemble contains all distinct sequences $x^{(k)}$ passing through a collection of pool states chosen at each time i . The ensemble is then updated to $y' = ((x^{(1)}, \theta'), \dots, (x^{(K)}, \theta'))$ using a Metropolis update that leaves ρ invariant. At this step, only θ is changed. We then map back to a new $x' = (x', \theta')$, where x' is now potentially different from the original x . We show that this method considerably improves sampling efficiency in the Ricker model of population dynamics.

As in the original Neal (2010) paper, we emphasize here that applications of ensemble methods are worth investigating when the density at each of the K elements of an ensemble can be computed in less time than it takes to do K separate density evaluations. For the stochastic volatility model, this is possible for ensembles over latent state sequences, and over the parameters c and σ^2 . In this paper, we will only consider joint ensembles over x and over σ . Since we will use $\eta = \log(\sigma^2)$ in the MCMC state, we will refer to ensembles over η below.

We propose two ensemble MCMC sampling schemes for the stochastic volatility model. The first, ENS1, updates the latent sequence, x , and η by mapping to an ensemble composed of latent sequences x and values of η , then immediately mapping back to new values of x and η . The second, ENS2, maps to an ensemble of latent state sequences x and values of η , like ENS1, then updates ϕ using an ensemble density summing over x and η , and finally maps back to new values of x and η .

For both ENS1 and ENS2, we first create a pool of η values with L_η elements, and at each time, i , a pool of values for the latent state x_i , with L_x elements. The current value of η is assigned to the pool element $\eta^{[1]}$ and for each time i , the current x_i is assigned to the pool element $x_i^{[1]}$. (Since the pool states are drawn independently, we don't need to randomly assign an index to the current η and the current x_i 's in their pools.) The remaining pool elements are drawn independently from some distribution having positive probability for all possible values of x_i and η , say κ_i for x_i and λ for η .

The total number of ensemble elements that we can construct using the pools over x_i and over η is $L_\eta L_x^N$. Naively evaluating the ensemble density presents an enormous computational burden for $L_x > 1$, taking time on the order of $L_\eta L_x^N$. By using the forward algorithm, together with

a “caching” technique, we can evaluate the ensemble density much more efficiently, in time on the order of $L_\eta L_x^2 N$. The forward algorithm is used to efficiently evaluate the densities for the ensemble over the x_i . The caching technique is used to efficiently evaluate the densities for the ensemble over η , which gives us a substantial constant factor speed-up in terms of computation time.

In detail, we do the following. Let $p(x_1)$ be the initial state distribution, $p(x_i|x_{i-1})$ the transition density for the latent process and $p(y_i|x_i, \eta)$ the observation probabilities. We begin by computing and storing the initial latent state probabilities — which do not depend on η — for each pool state $x_1^{[k]}$ at time 1.

$$P_1 = (p(x_1^{[1]}), \dots, p(x_1^{[L_x]})) \quad (17)$$

For each $\eta^{[l]}$ in the pool and each pool state $x_1^{[k]}$ we then compute and store the initial forward probabilities

$$\alpha_1^{[l]}(x_1^{[k]}|\eta^{[l]}) = p(x_1^{[k]}) \frac{p(y_1|x_1^{[k]}, \eta^{[l]})}{\kappa_1(x_1^{[k]})} \quad (18)$$

Then, for $i > 1$, we similarly compute and store the matrix of transition probabilities

$$P_i = \begin{pmatrix} p(x_i^{[1]}|x_{i-1}^{[1]}) & \dots & p(x_i^{[L_x]}|x_{i-1}^{[1]}) \\ \vdots & \ddots & \vdots \\ p(x_i^{[1]}|x_{i-1}^{[L_x]}) & \dots & p(x_i^{[L_x]}|x_{i-1}^{[L_x]}) \end{pmatrix}$$

where

$$p(x_i^{[k_1]}|x_{i-1}^{[k_2]}) \propto \exp(-(x_i^{[k_1]} - \phi x_{i-1}^{[k_2]})^2/2) \quad (19)$$

are transition probabilities between pool states $x_{i-1}^{[k_2]}$ and $x_i^{[k_1]}$ for $k_1, k_2 \in \{1, \dots, L_x\}$. We then use the stored values of the transition probabilities P_i to efficiently compute the vector of forward probabilities for all values of $\eta^{[l]}$ in the pool

$$\alpha_i^{[l]}(x_i|\eta^{[l]}) = \frac{p(y_i|x_i, \eta^{[l]})}{\kappa_i(x_i)} \sum_{k=1}^{L_x} p(x_i|x_{i-1}^{[k]}) \alpha_{i-1}^{[l]}(x_{i-1}^{[k]}|\eta^{[l]}), \quad i = 1, \dots, N \quad (20)$$

with $x_i \in \{x_i^{[1]}, \dots, x_i^{[L_x]}\}$.

At each time i , we divide the forward probabilities $\alpha_i^{[l]}(x_i)$ by $c_i^{[l]} = \sum_{k=1}^{L_x} \alpha_i^{[l]}(x_i|\eta^{[l]})$, storing the $c_i^{[l]}$ values and using the normalized $\alpha_i^{[l]}$'s in the next step of the recursion. This is needed to prevent underflow and for ensemble density computations. In the case of all the forward probabilities summing to 0, we set the forward probabilities at all subsequent times to 0. Note that we won't get underflows for all values of $\eta^{[l]}$, since we are guaranteed to have a log-likelihood that is not $-\infty$ for the current value of η in the MCMC state.

For each $\eta^{[l]}$, the ensemble density can then be computed as

$$\rho^{[l]} = \prod_{i=1}^N c_i^{[l]} \quad (21)$$

To avoid overflow or underflow, we work with the logarithm of $\rho^{[l]}$.

Even with caching, computing the forward probabilities for each $\eta^{[l]}$ in the pool is still an order L_x^2 operation since we multiply the vector of forward probabilities from the previous step by the transition matrix. However, if we do not cache and re-use the transition probabilities P_i when computing the forward probabilities for each value of $\eta^{[l]}$ in the pool, the computation of the ensemble densities $\rho^{[l]}$, for all $l = 1, \dots, L_\eta$, would be about 10 times slower. This is because computing forward probabilities for a value of η given saved transition probabilities only involves multiplications and additions, and not exponentiations, which are comparatively more expensive.

In ENS1, after mapping to the ensemble, we immediately sample new values of η and x from the ensemble. We first sample a $\eta^{[l]}$ from the marginal ensemble distribution, with probabilities proportional to $\rho^{[l]}$. After we have sampled an $\eta^{[l]}$, we sample a latent sequence x conditional on $\eta^{[l]}$, using a stochastic backwards recursion. The stochastic backwards recursion first samples a state x_N from the pool at time N with probabilities proportional to $\alpha_N^{[l]}(x_N|\eta^{[l]})$. Then, given the sampled value of x_i , we sample x_{i-1} from the pool at time $i - 1$ with probabilities proportional to $p(x_i|x_{i-1})\alpha_{i-1}^{[l]}(x_{i-1}|\eta^{[l]})$, going back to time 1.

In the terminology of Shestopaloff and Neal (2013) this is a “single sequence” update combined with an ensemble update for η (which is a “fast” variable in the terminology of Neal (2010) since recomputation of the likelihood function after changes to this variable is fast given the saved transition probabilities).

In ENS2, before mapping back to a new η and a new x as in ENS1, we perform a Metropolis update for ϕ using the ensemble density summing over all $\eta^{[l]}$ and all latent sequences in the ensemble, $\sum_{l=1}^{L_\eta} \rho^{[l]}$. This approximates updating ϕ using the posterior density of θ with x and η integrated out, when the number of pool states is large. The update nevertheless leaves the correct distribution exactly invariant, even if the number of pool states is not large.

2.1 Choosing the pool distribution

A good choice for the pool distribution is crucial for the efficient performance of the ensemble MCMC method.

For a pool distribution for x_i , a good candidate is the stationary distribution of x_i in the AR(1) latent process, which is $N(0, 1/\sqrt{1 - \phi^2})$. The question here is how to choose ϕ . For ENS1, which does not change ϕ , we can simply use the current value of ϕ from the MCMC state, call it ϕ_{cur} and draw pool states from $N \sim (0, c/\sqrt{1 - \phi_{\text{cur}}^2})$ for some scaling factor c . Typically, we would choose $c > 1$ in order to ensure that for different values of ϕ , we produce pool states that cover the region where x_i has high probability density.

We cannot use this pool selection scheme for ENS2 because the reverse transition after a change

in ϕ would use different pool states, undermining the proof via reversibility that the ensemble transitions leave the posterior distribution invariant. However, we can choose pool states that depend on both the current and the proposed values of ϕ , say ϕ and ϕ^* , in a symmetric fashion. For example, we can propose a value ϕ^* , and draw the pool states from $N \sim (0, c/\sqrt{1 - \phi_{\text{avg}}^2})$ where ϕ_{avg} is the average of ϕ and ϕ^* . The validity of this scheme can be seen by considering ϕ^* to be an additional variable in the model; proposing to update ϕ to ϕ^* can then be viewed as proposing to swap ϕ and ϕ^* within the MCMC state.

We choose pool states for η by sampling them from the model prior. Alternative schemes are possible, but we do not consider them here. For example, it is possible to draw local pool states for η which stay close to the current value of η by running a Markov chain with some desired stationary distribution J steps forwards and $L_\eta - J - 1$ steps backwards, starting at the current value of η . For details, see Neal (2003).

In our earlier work (Shestopaloff and Neal (2013)), one recommendation we made was to consider pool states that depend on the observed data y_i at a given point, constructing a “pseudo-posterior” for x_i using data observed at time i or in a small neighbourhood around i . For the ensemble updates ENS1 and ENS2 presented here, we cannot use this approach, as we would then need to make the pool states also depend on the current values of c and η , the latter of which is affected by the update. We could switch to the centered parametrization to avoid this problem, but that would prevent us from making η a fast variable.

3 Comparisons

The goal of our computational experiments is to determine how well the introduced variants of the ensemble method compare to our improved version of the Kastner and Fruwirth-Schnatter (2014) method. We are also interested in understanding when using a full ensemble update is helpful or not.

3.1 Data

We use a series simulated from the stochastic volatility model with parameters $c = 0.5, \phi = 0.98, \sigma^2 = 0.15$ with $N = 1000$. A plot of the data is presented in Figure 1.

We use the following priors for the model parameters.

$$c \sim N(0, 1) \tag{22}$$

$$\phi \sim \text{Unif}[0, 1] \tag{23}$$

$$\sigma^2 \sim \text{Inverse-Gamma}(2.5, 0.075) \tag{24}$$

We use the parametrization in which the Inverse-Gamma(α, β) has probability density

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta/x}, \quad x > 0 \tag{25}$$

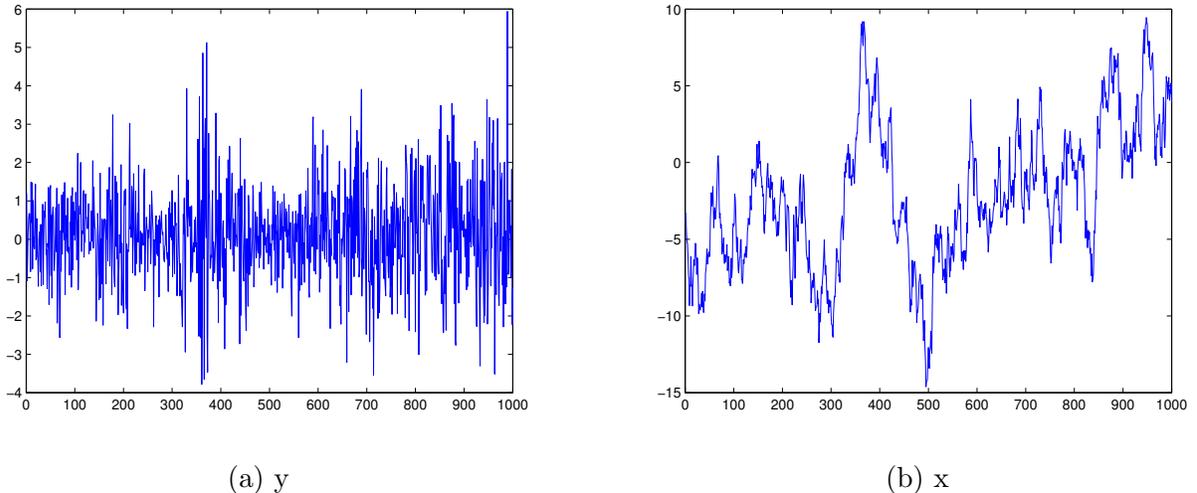


Figure 1: Data set used for testing.

For $\alpha = 2.5, \beta = 0.075$ the 2.5% and 97.5% quantiles of this distribution are approximately (0.0117, 0.180).

In the MCMC state, we transform ϕ and σ^2 to

$$\eta = \log(\sigma^2) \tag{26}$$

$$\gamma = \log((1 + \phi)/(1 - \phi)) \tag{27}$$

with the priors transformed correspondingly.

3.2 Sampling schemes and tuning

We compare three sampling schemes — the Kastner and Fruwirth-Schnatter (KF) method, and our two ensemble schemes, ENS1, in which we map to an ensemble of η and x values and immediately map back, and ENS2, in which we additionally update γ with an ensemble update before mapping back.

We combine the ensemble scheme with the computationally cheap ASIS Metropolis updates. It is sensible to add cheap updates to a sampling scheme if they are available. Note that the ASIS (or translation and scale) updates we use in this paper are generally applicable to location-scale models and are not restricted by the linear and Gaussian assumption.

Pilot runs showed that 80 updates appears to be the point at which we start to get diminishing returns from using more Metropolis updates (given the sufficient statistics) in the KF scheme. This is the number of Metropolis updates we use with the ensemble schemes as well.

The KF scheme updates the state as follows:

1. Update x , using the Kalman filter-based update, using the current mixture indicators r .

2. Update the parameters using the mixture approximation to the observation density. This step consists of 80 Metropolis updates to ϕ given the sufficient statistics for NC, followed by one joint update of c and η .
3. Change to the C parametrization.
4. Update all three parameters simultaneously using 80 Metropolis updates, given the sufficient statistics for C. Note that this update does not depend on the observation density and is therefore exact.
5. Update the mixture indicators r .

The ENS1 scheme proceeds as follows:

1. Map to an ensemble of η and x .
2. Map back to a new value of η and x .
3. Do steps 2) - 4) as for KF, but with the exact observation density.

The ENS2 scheme proceeds as follows:

1. Map to an ensemble of η and x .
2. Update γ using an ensemble Metropolis update.
3. Map back to a new value of η and x .
4. Do steps 2) - 4) as for KF, but with the exact observation density.

The Metropolis updates use a normal proposal density centered at the current parameter values. Proposal standard deviations for the Metropolis updates in NC were set to estimated marginal posterior standard deviations, and to half of that in C. This is because in C, we update all three parameters at once, whereas in NC we update c and η jointly and ϕ separately. The marginal posterior standard deviations were estimated using a pilot run of the ENS2 method. The tuning settings for the Metropolis updates are presented in Table 1.

For ensemble updates of γ , we also use a normal proposal density centered at the current value of γ , with a proposal standard deviation of 1, which is double the estimated marginal posterior standard deviation of γ . The pool states over x_i are selected from the stationary distribution of the AR(1) latent process, with standard deviation $2/\sqrt{1 - \phi_{\text{cur}}^2}$ for the ENS1 scheme and $2/\sqrt{1 - \phi_{\text{avg}}^2}$ for the ENS2 scheme. We used the prior density of η to select pool states for η .

For each method, we started the samplers from 5 randomly chosen points. Parameters were initialized to their prior means (which were 0 for c , 1.39 for γ and -3.29 for η), and each $x_i, i = 1, \dots, N$, was initialized independently to a value randomly drawn from the stationary distribution of the AR(1) latent process, given γ set to the prior mean. For the KF updates, the mixture indicators r were all initialized to 5's, this corresponds to the mixture component whose median matches the median of the $\log(\chi_1^2)$ distribution most closely. All methods were run for approximately the same amount of computational time.

3.3 Results

Before comparing the performance of the methods, we verified that the methods give the same answer up to expected variation by looking at the 95% confidence intervals each produced for the posterior means of the parameters. These confidence intervals were obtained from the standard error of the average posterior mean estimate over the five runs. The KF estimates were adjusted using the importance weights that compensate for the use of the approximate observation distribution. No significant disagreement between the answers from the different methods was apparent. We then evaluated the performance of each method using estimates of autocorrelation time, which measures how many MCMC draws are needed to obtain the equivalent of one independent draw.

To estimate autocorrelation time, we first estimated autocovariances for each of the five runs, discarding the first 10% of the run as burn-in, and plugging in the overall mean of the five runs into the autocovariance estimates. (This allows us to detect if the different runs for each method are exploring different regions of the parameter/latent variable space). We then averaged the resulting autocovariance estimates and used this average to get autocorrelation estimates $\hat{\rho}_k$. Finally, autocorrelation time was estimated as $1 + 2 \sum_{k=1}^K \hat{\rho}_k$, with K chosen to be the point beyond which the ρ_k become approximately 0. All autocovariances were estimated using the Fast Fourier Transform for computational efficiency.

The results are presented in Tables 2 and 3. The timings for each sampler represent an average over 100 iterations (each iteration consisting of the entire sequence of updates), with the samplers started from a point taken after the sampler converged to equilibrium. The program was written in MATLAB and run on a Linux system with an Intel Xeon X5680 3.33 GHz CPU. For a fair comparison, we multiply estimated autocorrelation times by the time it takes to do one iteration and compare these estimates.

We ran the KF method for 140,000 iterations, with estimated autocorrelation times using the original (unweighed) sequence for (c, γ, η) of $(2.1, 37, 73)$, which after adjusting by computation time of 0.16 seconds per iteration are $(0.34, 5.9, 12)$. It follows that the ENS1 method with L_x set to 50 and L_η set to 10 is better than the KF method by a factor of about 3.1 for the parameter η . For ENS2, the same settings $L_x = 50$ and $L_\eta = 10$ appears to give the best results, with ENS2 worse by a factor of about 1.7 than ENS1 for sampling η . We also see that the ENS1 and ENS2 methods aren't too sensitive to the particular tuning parameters, so long as there is a sufficient number of ensemble elements both for x_i and for η .

Method	Prop. Std. (NC)			Acc. Rate for γ (NC)	Acc. Rate for (c, η) (NC)	Prop. Std. (C)			Acc. Rate for (c, γ, η)
	c	γ	η			c	γ	η	
KF					0.12				
ENS1	0.21	0.5	0.36	0.52	0.12	0.105	0.25	0.18	0.22
ENS2									

Table 1: Metropolis proposal standard deviations with associated acceptance rates.

L_x	L_η	Iterations	Time/iter (s)	ACT			ACT \times time		
				c	γ	η	c	γ	η
10	1	195000	0.11	2.6	99	160	0.29	11	18
	10	180000	0.12	2.7	95	150	0.32	11	18
	30	155000	0.14	2.6	81	130	0.36	11	18
	50	140000	0.16	2.3	91	140	0.37	15	22
30	1	155000	0.14	2.4	35	71	0.34	4.9	9.9
	10	135000	0.16	2.2	18	26	0.35	2.9	4.2
	30	110000	0.20	2.3	19	26	0.46	3.8	5.2
	50	65000	0.33	1.9	16	24	0.63	5.3	7.9
50	1	115000	0.19	1.9	34	68	0.36	6.5	13
	10	95000	0.23	1.9	11	17	0.44	2.5	3.9
	30	55000	0.38	2.2	8.9	12	0.84	3.4	4.6
	50	55000	0.39	1.9	11	14	0.74	4.3	5.5
70	1	85000	0.25	2.2	33	67	0.55	8.3	17
	10	60000	0.38	1.9	8.3	11	0.72	3.2	4.2
	30	50000	0.42	1.8	8.4	11	0.76	3.5	4.6
	50	45000	0.48	1.9	9.1	12	0.91	4.4	5.8

Table 2: Performance of method ENS1.

L_x	L_η	Acc. Rate for γ	Iterations	Time/iter (s)	ACT			ACT \times time		
					c	ϕ	η	c	γ	η
10	1	0.32	110000	0.20	2.5	100	170	0.5	20	34
	10	0.32	95000	0.23	2.4	91	140	0.55	21	32
	30	0.32	80000	0.27	2.5	97	150	0.68	26	41
	50	0.32	70000	0.30	2.7	90	140	0.81	27	42
30	1	0.33	80000	0.26	2.3	34	68	0.6	8.8	18
	10	0.33	70000	0.31	2.3	18	26	0.71	5.6	8.1
	30	0.33	55000	0.39	2	18	27	0.78	7	11
	50	0.34	35000	0.61	2.4	12	19	1.5	7.3	12
50	1	0.34	60000	0.36	1.7	33	69	0.61	12	25
	10	0.35	50000	0.44	2.1	10	15	0.92	4.4	6.6
	30	0.34	30000	0.71	1.8	10	15	1.3	7.1	11
	50	0.34	25000	0.81	1.8	12	17	1.5	9.7	14
70	1	0.34	45000	0.49	2.2	29	61	1.1	14	30
	10	0.35	30000	0.72	1.6	7.3	11	1.2	5.3	7.9
	30	0.36	25000	0.86	1.6	7.3	9.3	1.4	6.3	8
	50	0.36	25000	0.96	1.8	5.9	7.8	1.7	5.7	7.5

Table 3: Performance of method ENS2.

Method	c	γ	η
KF	0.2300 (\pm 0.0004)	4.3265 (\pm 0.0053)	-3.7015 (\pm 0.0054)
ENS1	0.2311 (\pm 0.0006)	4.3228 (\pm 0.0017)	-3.6986 (\pm 0.0015)
ENS2	0.2306 (\pm 0.0008)	4.3303 (\pm 0.0025)	-3.7034 (\pm 0.0021)

Table 4: Estimates of posterior means, with standard errors of posterior means shown in brackets.

The results show that using a small ensemble (10 or so pool states) over η is particularly helpful. One reason for this improvement is the ability to use the caching technique to make these updates computationally cheap. A more basic reason is that updates of η consider the entire collection of latent sequences, which allows us to make large changes to η , compared to the Metropolis updates.

Even though the ENS2 method in this case is outperformed by the ENS1 method, we have only applied it to one data set and there is much room for further tuning and improvement of the methods. A possible explanation for the lack of substantial performance gain with the ensemble method is that conditional on a single sequence, the distribution of ϕ has standard deviation comparable to its marginal standard deviation, which means that we can't move too much further with an ensemble update than we do with our Metropolis updates. An indication of this comes from the acceptance rate for ensemble updates of γ in ENS2, which we can see isn't improved by much as more pool states are added.

Parameter estimates for the best performing KF, ENS1 and ENS2 settings are presented in Table 4. These estimates were obtained by averaging samples from all 5 runs with 10% of the sample discarded as burn-in. We see that the differences between the standard errors are in approximate agreement with the differences in autocorrelation times for the different methods.

4 Conclusion

We found that noticeable performance gains can be obtained by using ensemble MCMC based sampling methods for the stochastic volatility model. It may be possible to obtain even larger gains on different data sets, and with even better tuning. In particular, it is possible that the method of updating ϕ with an ensemble, or some variation of it, actually performs better than a single sequence method in some other instance.

The method of Kastner and Fruwirth-Schnatter (2014) relies on the assumption that the state process is linear and Gaussian, which enables efficient state sequence sampling using Kalman filters. The method would not be applicable if this was not the case. However, the ensemble method could still be applied to this case as well. It would be of interest to investigate the performance of ensemble methods for stochastic volatility models with different noise structures for the latent process. It would also be interesting to compare the performance of the ensemble MCMC method with the PMCMC-based methods of Andrieu et. al (2010) and also to see whether techniques used to improve PMCMC methods can be used to improve ensemble methods and vice versa.

Multivariate versions of stochastic volatility models, for example those considered in Scharth and Kohn (2013) are another class of models for which inference is difficult, and that it would be interesting to apply the ensemble MCMC method to. We have done preliminary experiments applying ensemble methods to multivariate stochastic volatility models, with promising results. For these models, even though the latent process is linear and Gaussian, due to a non-constant covariance matrix the observation process does not have a simple and precise mixture of Gaussians approximation.

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Appendix

Here, we derive the sufficient statistics for the stochastic volatility model in the two parametrizations and the likelihoods in terms of sufficient statistics.

For NC, we derive low-dimensional sufficient statistics for ϕ as follows

$$\begin{aligned}
p(x|\phi) &\propto \sqrt{1-\phi^2} \exp\left(- (1-\phi^2)x_1^2/2\right) \exp\left(-\sum_{i=2}^N (x_i - \phi x_{i-1})^2/2\right) \\
&\propto \sqrt{1-\phi^2} \exp\left(- (x_1^2 - \phi^2 x_1^2 + \sum_{i=2}^N x_i^2 - 2\phi \sum_{i=2}^N x_i x_{i-1} + \phi^2 \sum_{i=2}^N x_{i-1}^2)/2\right) \\
&\propto \sqrt{1-\phi^2} \exp\left(- (\phi^2 \sum_{i=2}^{N-1} x_i^2 - 2\phi \sum_{i=2}^N x_i x_{i-1} + \sum_{i=1}^N x_i^2)/2\right)
\end{aligned}$$

Letting

$$t_1 = \sum_{i=1}^N x_i^2, \quad t_2 = \sum_{i=2}^N x_{i-1} x_i, \quad t_3 = x_1^2 + x_N^2$$

we can write

$$p(x|\phi) \propto \sqrt{1-\phi^2} \exp(-(\phi^2(t_1 - t_3) - 2\phi t_2 + t_1)/2)$$

For C, we have

$$\begin{aligned}
p(x|c, \phi, \sigma^2) &\propto \frac{\sqrt{1-\phi^2}}{\sigma^{n/2}} \exp\left(- (1-\phi^2)(\tilde{x}_1 - c)^2/2\sigma^2\right) \exp\left(- \sum_{i=2}^N ((\tilde{x}_i - (c + \phi(\tilde{x}_{i-1} - c)))^2/2\sigma^2)\right) \\
&\propto \frac{\sqrt{1-\phi^2}}{\sigma^{n/2}} \exp\left(- (\tilde{x}_1^2 - \phi^2\tilde{x}_1^2 - 2\tilde{x}_1c + c^2 + 2c\phi^2\tilde{x}_1 - c^2\phi^2 + \sum_{i=2}^N \tilde{x}_i^2 \right. \\
&\quad \left. - 2\sum_{i=2}^N \tilde{x}_i(c + \phi(\tilde{x}_{i-1} - c)) + \sum_{i=2}^N (c + \phi(\tilde{x}_{i-1} - c))^2/2\sigma^2)\right) \\
&\propto \frac{\sqrt{1-\phi^2}}{\sigma^{n/2}} \exp\left(- \left(\sum_{i=1}^N \tilde{x}_i^2 - \phi^2\tilde{x}_1^2 - 2\tilde{x}_1c + c^2 + 2c\phi^2\tilde{x}_1 - c^2\phi^2 - 2c\sum_{i=2}^N \tilde{x}_i \right. \right. \\
&\quad \left. \left. - 2\phi\sum_{i=2}^N \tilde{x}_i\tilde{x}_{i-1} + 2c\phi\sum_{i=2}^N \tilde{x}_i + (N-1)c^2 + 2c\phi\sum_{i=2}^N (\tilde{x}_{i-1} - c) + \phi^2\sum_{i=2}^N (\tilde{x}_{i-1} - c)^2/2\sigma^2\right)\right) \\
&\propto \frac{\sqrt{1-\phi^2}}{\sigma^{n/2}} \exp\left(- \left(\sum_{i=1}^N \tilde{x}_i^2 - \phi^2\tilde{x}_1^2 - 2\tilde{x}_1c + c^2 + 2c\phi^2\tilde{x}_1 - c^2\phi^2 - 2c\sum_{i=2}^N \tilde{x}_i \right. \right. \\
&\quad \left. \left. - 2\phi\sum_{i=2}^N \tilde{x}_i\tilde{x}_{i-1} + 2c\phi\sum_{i=2}^N \tilde{x}_i + (N-1)c^2 + 2c\phi\sum_{i=2}^N \tilde{x}_{i-1} - 2(N-1)c^2\phi \right. \right. \\
&\quad \left. \left. + \phi^2\sum_{i=2}^N \tilde{x}_{i-1}^2 - 2c\phi^2\sum_{i=2}^N \tilde{x}_{i-1} + (N-1)c^2\phi^2/2\sigma^2\right)\right) \\
&\propto \frac{\sqrt{1-\phi^2}}{\sigma^{n/2}} \exp\left(- \left(\sum_{i=1}^N \tilde{x}_i^2 + \phi^2\sum_{i=2}^{N-1} \tilde{x}_i^2 - 2\phi\sum_{i=2}^N \tilde{x}_{i-1}\tilde{x}_i + 2c\phi^2\sum_{i=2}^{N-1} \tilde{x}_i - 2c\sum_{i=1}^N \tilde{x}_i \right. \right. \\
&\quad \left. \left. + 4c\phi\sum_{i=2}^{N-1} \tilde{x}_i + 2c\phi(\tilde{x}_1 + \tilde{x}_N) + (N-1)(c(\phi-1))^2 + c^2(1-\phi^2)/2\sigma^2\right)\right)
\end{aligned}$$

Letting

$$\tilde{t}_1 = \sum_{i=1}^N \tilde{x}_i^2, \quad \tilde{t}_2 = \sum_{i=2}^{N-1} \tilde{x}_i^2, \quad \tilde{t}_3 = \sum_{i=2}^N \tilde{x}_{i-1}\tilde{x}_i, \quad \tilde{t}_4 = \sum_{i=2}^{N-1} \tilde{x}_i, \quad \tilde{t}_5 = \tilde{x}_1 + \tilde{x}_N$$

we can write

$$\begin{aligned}
p(x|c, \phi, \sigma^2) &= \frac{\sqrt{1-\phi^2}}{\sigma^{n/2}} \exp\left(- (\tilde{t}_1 + \phi^2\tilde{t}_2 - 2\phi\tilde{t}_3 - 2c\phi^2\tilde{t}_4 - 2c(\tilde{t}_4 + \tilde{t}_5) \right. \\
&\quad \left. + 4c\phi\tilde{t}_4 + 2c\phi\tilde{t}_5 + (N-1)(c(\phi-1))^2 + c^2(1-\phi^2)/2\sigma^2\right)
\end{aligned}$$