Question 1: The random variable $X$ has a range of $\{0, 1, 2\}$ and the random variable $Y$ has a range of $\{1, 2\}$. The joint distribution of $X$ and $Y$ is given by the following table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$P(X = x, Y = y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0.1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.2</td>
</tr>
</tbody>
</table>

1. Write down tables for the marginal distributions of $X$ and of $Y$, i.e. give the values of $P(X = x)$ for all $x$, and of $P(Y = y)$ for all $y$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(X = x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$y$</th>
<th>$P(Y = y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
</tr>
</tbody>
</table>

2. Write down a table for the conditional distribution of $X$ given that $Y = 2$, i.e. give the values of $P(X = x \mid Y = 2)$ for all $x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(X = x \mid Y = 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
</tr>
</tbody>
</table>

3. Compute $E(X)$ and $E(Y)$.

$$E(X) = 0 \times 0.3 + 1 \times 0.2 + 2 \times 0.5 = 1.2$$
$$E(Y) = 1 \times 0.5 + 2 \times 0.5 = 1.5$$

4. Compute $E(XY)$.

$$E(XY) = (0 \times 1) \times 0.2 + (0 \times 2) \times 0.1 + (1 \times 1) \times 0.0 + (1 \times 2) \times 0.2 + (2 \times 1) \times 0.3 + (2 \times 2) \times 0.2 = 1.8$$
5. Are \( X \) and \( Y \) independent? Explain why or why not.

No, they are not independent, as can be seen from the fact that \( P(X = 1, Y = 1) \neq P(X = 1)P(Y = 1) \). (The left side is 0, but the right side is \( 0.2 \times 0.5 = 0.1 \).)

Note that the fact that \( E(XY) = E(X)E(Y) \) does NOT imply that \( X \) and \( Y \) are independent. \( E(XY) = E(X)E(Y) \) is implied by \( X \) and \( Y \) being independent, but not the other way around.

Question 2: You roll one red die and one green die. Define the random variables \( X \) and \( Y \) as follows:

\[
X = \text{The number showing on the red die} \\
Y = \text{The number of dice that show the number six}
\]

For example, if the red and green dice show the numbers 6 and 4, then \( X = 6 \) and \( Y = 1 \).

Write down a table showing the joint probability mass function for \( X \) and \( Y \), find the marginal distribution for \( Y \), and compute \( E(Y) \).

Here is a table showing the joint probability mass function, with the marginal distribution for \( Y \) on the right:

\[
\begin{array}{c|ccccccc}
& 1 & 2 & 3 & 4 & 5 & 6 & \\
\hline
X = & 0 & 5/36 & 5/36 & 5/36 & 5/36 & 5/36 & 0 \\
Y = 1 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 5/36 & 10/36 \\
2 & 0 & 0 & 0 & 0 & 0 & 1/36 & 1/36 \\
\end{array}
\]

The expectation of \( Y \) is

\[
E(Y) = 0 \times \frac{25}{36} + 1 \times \frac{10}{36} + 2 \times \frac{1}{36} = \frac{12}{36} = \frac{1}{3}
\]

Question 3. Suppose you roll two fair, six-sided dice, one of which is red and the other of which is green. Define the following random variables:

\[
X = \text{The number shown on the red die} \\
Y = \begin{cases} 
0 & \text{if the two dice show the same number} \\
1 & \text{if the number on the green die is bigger than the number on the red die} \\
2 & \text{if the number on the red die is bigger than the number on the green die}
\end{cases}
\]

a) Write down a table showing the joint probability mass function for \( X \) and \( Y \).

\[
\begin{array}{c|ccccccc}
& 1 & 2 & 3 & 4 & 5 & 6 & \\
\hline
X = & 0 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 \\
Y = 1 & 5/36 & 4/36 & 3/36 & 2/36 & 1/36 & 0 \\
2 & 0 & 1/36 & 2/36 & 3/36 & 4/36 & 5/36 & \\
\end{array}
\]
b) Find the marginal probability mass function for \( Y \), and compute its expected value.

\[
\begin{array}{c|c}
 y & P(Y = y) \\
0 & 6/36 \\
1 & 15/36 \\
2 & 15/36
\end{array}
\]

\[
E(Y) = 0 \times (6/36) + 1 \times (15/36) + 2 \times (15/36) = 45/36
\]

c) Find the conditional probability mass function for \( X \) given \( Y = 1 \).

\[
X =
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

Question 4. You flip a fair coin. If the coin lands heads, you roll a fair six-sided die 100 times. If the coin lands tails, you roll the die 101 times. Let \( X \) be 1 if the coin lands heads and 0 if the coin lands tails. Let \( Y \) be the total number of times that you roll a 6. Find \( P(X = 1 \mid Y = 15) \).

We can find \( P(X = 1 \mid Y = 15) \) using Bayes’ Rule. It’s easiest using the “odds” form:

\[
\frac{P(X = 1 \mid Y = 15)}{P(X = 0 \mid Y = 15)} = \frac{P(X = 1)}{P(X = 0)} \times \frac{P(Y = 15 \mid X = 1)}{P(Y = 15 \mid X = 0)}
\]

The ratio \( P(X = 1) / P(X = 0) \) is one, since heads and tails are equally likely.

The conditional probabilities \( P(Y = 15 \mid X = 1) \) and \( P(Y = 15 \mid X = 0) \) are both binomial. Given \( X = 1 \), \( Y \) has the binomial distribution with \( n = 100 \) and \( p = 1/6 \), so

\[
P(Y = 15 \mid X = 1) = \left( \begin{array}{c}
100 \\
15
\end{array} \right) (1/6)^{15} (1-1/6)^{100-15}
\]

Given \( X = 0 \), \( Y \) has the binomial distribution with \( n = 101 \) and \( p = 1/6 \).

\[
P(Y = 15 \mid X = 0) = \left( \begin{array}{c}
101 \\
15
\end{array} \right) (1/6)^{15} (1-1/6)^{101-15}
\]

The odds for \( X = 1 \) given \( Y = 15 \) are therefore

\[
\frac{P(X = 1 \mid Y = 15)}{P(X = 0 \mid Y = 15)} = 1 \times \frac{\left( \begin{array}{c}
100 \\
15
\end{array} \right) (1/6)^{15} (1-1/6)^{100-15}}{\left( \begin{array}{c}
101 \\
15
\end{array} \right) (1/6)^{15} (1-1/6)^{101-15}} = \frac{86}{101} \times \frac{6}{5} = \frac{516}{505}
\]

which gives \( P(X = 1 \mid Y = 15) = (516/505) / (1 + 516/505) = 0.505 \ldots \). (See the week 3 lecture summary regarding odds.)
**Question 5.** Suppose you randomly pick an integer from 1 to 3, with the three possibilities being equally likely. Call this integer $N$. You then randomly pick an integer from $N$ to 3, with the $4 - N$ possibilities being equally likely. Call this second integer $M$. What is the probability that $M$ will be 3?

\[
P(M = 3) = \sum_{n=1}^{3} P(M = 3, N = n)
\]

\[
= \sum_{n=1}^{3} P(M = 3 \mid N = n) P(N = n)
\]

\[
= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) = \frac{11}{18}
\]

**Question 6.** Three computers, A, B, and C, are linked by network connections as shown below:

Two network connections, a1 and a2, link computer A and computer B. Three network connections, b1, b2, and b3, link computer B and computer C. Since there is no direct connection from computer A to computer C, messages sent from computer A to computer C must pass through computer B.

When computer A sends a message to computer B, it randomly chooses whether to use connection a1 or connection a2, with probabilities of 2/3 for a1 and 1/3 for a2. When computer B sends a message to computer C, it randomly chooses whether to use connection b1, connection b2, or connection b3, with probabilities of 1/2 for b1, 1/4 for b2, and 1/4 for b3.

The time to send a message through each connection is 1ms for a1, 2ms for a2, 1ms for b1, 2ms for b2, and 3ms for b3. When a message is sent from computer A to computer C through computer B, it takes no time for computer B to take the message received from connection a1 or a2 and send it on connection b1, b2, or b3.

Let $X$ be the random variable whose value is the total time (in milliseconds) taken for a message sent from computer A to computer C to arrive.

a) Find $P(X = 4)$.

\[
P(X = 4) = P(\text{takes path a1, b3 or path a2, b2})
\]

\[
= P(\text{takes path a1, b3}) + P(\text{takes path a2, b2})
\]

\[
= (2/3)(1/4) + (1/3)(1/4) = 1/4
\]
b) Find $E(X)$.

This can be found by finding the probabilities for all values of $X$, similarly to what is done above for part (a), and then applying the definition of expectation.

It can more easily be solved by letting $U$ be the time from A to B and $V$ be the time from B to C, so that $X = U + V$. Then $E(X) = E(U) + E(V)$. We can compute that $E(U) = 1 \times (2/3) + 2 \times (1/3) = 4/3$ and $E(V) = 1 \times (1/2) + 2 \times (1/4) + 3 \times (1/4) = 7/4$. So $E(X) = (4/3) + (7/4) = 37/12$.

c) Suppose that a message sent from computer A to computer C takes 4ms or more to arrive (ie, $X \geq 4$). How likely is it that this message was sent from computer B to computer C using connection b3?

$X \geq 4$ when the path taken is $a1, b3$ or $a2, b2$ or $a2, b3$. Of these, the paths $a1, b3$ and $a2, b3$ involve connection b3. The conditional probability of b3 having been used given that $X \geq 4$ is therefore

$$P(path \text{ is } a1, b3 \text{ or } a2, b3) = \frac{(2/3)(1/4) + (1/3)(1/4)}{(2/3)(1/4) + (1/3)(1/4) + (1/3)(1/4)} = \frac{3}{4}$$