STA 3000, Fall 2008 — Assignment #2

Due November 20, at start of lecture. Worth 8% of the course grade.

This assignment is to be done by each student individually. You may discuss it in general terms with other students, but the work you hand in should be your own. In particular, you should not leave any discussion of this assignment with any written notes or other recordings, nor receive any written or other material from anyone (except your instructor) by other means such as email.

For all questions, show both the final answer and how you obtained it.

Question 1: We observe the values of X and Y, which are independent given values for a real parameter μ and a positive real parameter σ. The distribution of X is $N(\mu, \sigma^2)$. The distribution of Y is $N(a\mu, (b\sigma)^2)$, where a and b are known positive constants.

1. Express this model in exponential family form, find the natural sufficient statistic, and identify the natural parameter in terms of μ and σ.
2. Show how the expected value of the natural sufficient statistic can be found by differentiating the normalizing factor for the model. Using this result, identify the mean parameters of the model in terms of μ and σ.
3. Find the maximum likelihood estimates for the mean parameters, for the natural parameters, and for the original parameters μ and σ.

Question 2: Triples (X, Y, Z) are generated from a process in which X, Y, and Z are independent (given the values of unknown parameters α, β, γ), with $X \sim \text{Poisson}(\alpha)$, $Y \sim \text{Poisson}(\beta)$, and $Z \sim \text{Poisson}(\gamma)$. We do not observe all such triples, however. Instead, we see only those triples for which $X + Y + Z = c$, for some known constant c (a positive integer). Suppose we obtain n such triples, $(X_1, Y_1, Z_1), \ldots, (X_n, Y_n, Z_n)$. Show that the model for this observed data is an exponential family, and express it in a minimal form (ie, which is not degenerate with respect to either the natural parameters or the natural sufficient statistic).

Question 3: Let $F_1(\theta_1)$ and $F_2(\theta_2)$ be exponential family models, in which $\theta_1 \in \mathrm{R}$ and $\theta_2 \in \mathrm{R}$ are the natural parameters, and in which the data spaces are the reals or subsets of the reals. Define a model for (X, Y) that has parameters $\phi_1 \in \mathrm{R}$ and $\phi_2 \in \mathrm{R}$, with $X|\phi_1, \phi_2 \sim F_1(\phi_1)$ and $Y|X = x, \phi_1, \phi_2 \sim F_2(x\phi_2)$. Investigate when this is an exponential family model. Specifically,

1. Find a specific example for which this model for (X, Y) **is not** an exponential family model.
2. Find a specific example for which this model for (X, Y) **is** an exponential family model.
3. State and prove a theorem that says this model for (X, Y) is an exponenental family model provided some condition is met, making this condition as general as you can.