Study question set #1 for STA 3000, Spring 2014

These are for study only, not to hand in for credit

Question 1:

Given $\theta \in (1, \infty)$, observations $Y_1, \ldots, Y_n \overset{\text{IID}}{\sim} U(0, \theta)$. Suppose that we use as the prior for θ the distribution with density proportional to $I(\theta > 1)/\theta^2$.

A) Find the posterior density for θ given $y_1, \ldots, y_n$.

B) Find the predictive density for a new observation, Y_{n+1}, given $y_1, \ldots, y_n$. Assume that, given a value for θ, Y_{n+1} is independent of $Y_1, \ldots, Y_n$, and has the same distribution as each of $Y_1, \ldots, Y_n$.

Question 2: Consider a decision problem in which the data space, parameter space, and action space are all the set of real numbers. The distribution of the data, X, given the parameter, θ, is $N(\theta, 1)$. The loss function is $L(\theta, a) = (a - \theta)^2$.

Consider decision rules of the form $\delta_{a,b}(x) = ax + b$, where a and b can be any real numbers.

A) Find a simple formula for the risk function, $R(\theta, \delta_{a,b})$ for all a and b.

B) Which rules $\delta_{a,b}$ are dominated by some other rule of this form (ie, by $\delta_{a',b'}$ for some a' and b')?

C) For each of the rules of this form that are not dominated by another of this form, can you find a prior distribution for θ for which this rule is a formal Bayes rule?

D) Say what you can about the admissibility of each of the rules $\delta_{a,b}$.

Question 3: Consider a decision problem with data $X = (X_1, X_2, X_3)$, in which X_1, X_2, and X_3 are independent given the value of a parameter $\Theta \in \{0, 1, 2, 3\}$, with

$$X_i | \Theta = \theta \ \sim \ \text{Poisson}(1 + I_{\{i\}}(\theta))$$

In other words, if $\Theta = 0$, all X_i are i.i.d. from the Poisson(1) distribution. Otherwise, X_θ is from the Poisson(2) distribution and the other X_i are from the Poisson(1) distribution. After observing X, we take an action $a \in \{0, 1, 2, 3\}$, and suffer loss given by

$$L(\theta, a) = \begin{cases} 0 & \text{if } a = \theta \\ 1 & \text{if } a = 0 \text{ and } \theta \neq 0 \\ 2 & \text{if } a \neq 0 \text{ and } \theta \neq a \end{cases}$$

Suppose our prior distribution for Θ is

$$P(\Theta = 0) = 1/2, \quad P(\Theta = 1) = P(\Theta = 2) = P(\Theta = 3) = 1/6$$

Find a Bayes rule for this decision problem. Say explicitly what decision should be taken according to this rule for each of the following data sets, (X_1, X_2, X_3):

$$(4, 0, 0), \ \ (5, 0, 1), \ \ (0, 5, 6), \ \ (0, 4, 5), \ \ (5, 6, 5)$$

Question 4: Consider a decision problem with data space $\{1,2\}$, parameter space $\{\theta_0, \theta_1\}$, and action space $\{a_0, a_1\}$. Suppose that the loss function is

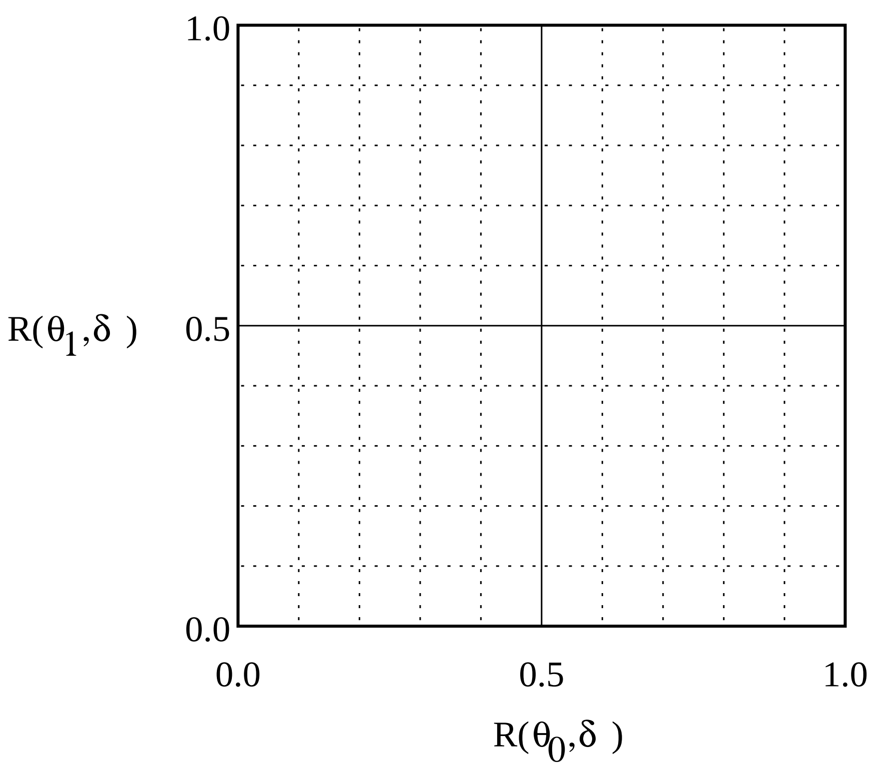
$$L(\theta, a) = \begin{cases} 0.2 & \text{if } \theta = \theta_0 \text{ and } a = a_0 \\ 0.8 & \text{if } \theta = \theta_0 \text{ and } a = a_1 \\ 0.7 & \text{if } \theta = \theta_1 \text{ and } a = a_0 \\ 0.4 & \text{if } \theta = \theta_1 \text{ and } a = a_1 \end{cases}$$

and that the data distributions are given by

$$P(y|\theta_0) = \begin{cases} 0.5 & \text{if } y = 1 \\ 0.5 & \text{if } y = 2 \end{cases}$$

$$P(y|\theta_1) = \begin{cases} 0.0 & \text{if } y = 1 \\ 1.0 & \text{if } y = 2 \end{cases}$$

A) Draw the risk set for this decision problem in the diagram below:



B) Find a minimax rule for this decision problem. (Write an explicit formula for the decision rule.) Is this the unique minimax rule? Is it admissible? Explain your answers (geometric explanations are OK).

C) Find a Bayes rule for this decision problem when the prior gives equal probability to θ_0 and θ_1. (Write an explicit formula for the decision rule.) Is this Bayes rule unique? Is it admissible? Explain.

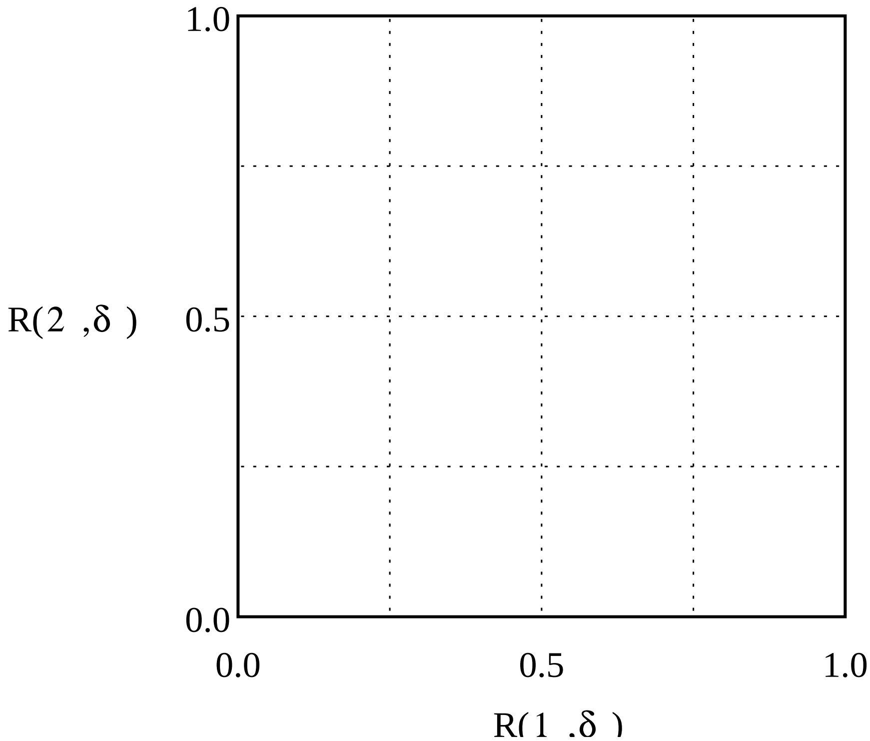
D) Are all Bayes rules for this decision problem (for all possible priors) admissible? Explain.

Question 5: Consider two pairs of data points, (X_1, Y_1) and (X_2, Y_2), with $(X_i, Y_i) \in \{1, 2, 3\}^2$. We model these pairs using a parameter $\theta \in \{1, 2\}$. Given θ, the two pairs are independent. If $\theta = 2$, the distribution of each is uniform over the data space, and if $\theta = 1$, the distribution of each is an equal mixture of this uniform distribution and the uniform distribution over the subset where $X_i = Y_i$. In other words,

$$(X_1, Y_1),\ (X_2, Y_2) \mid \theta \overset{\text{IID}}{\sim} (\theta/2)\,\text{Uniform}(\{1, 2, 3\}^2) \ +\ (1 - \theta/2)\,\text{Uniform}(\{(1,1),\ (2,2),\ (3,3)\})$$

We wish to estimate θ from the observed (x_1, y_1) and (x_2, y_2), with loss function $L(\theta, a) = |\theta - a|$, where a is in the action space $\{1, 2\}$.

A) On the diagram below, plot the risk set for this decision problem — ie, the set of risk functions that can be obtained from a randomized or non-randomized decision rule. Explain how this can be done without explicitly considering all 2^{81} = 2417851639229258349412352 non-randomized decision rules.



B) Find two non-randomized Bayes rules for this decision problem that are admissible, and that have different risk functions.

C) Are any of the Bayes rules for this decision problem inadmissible? If so, give one.

D) Find a minimax rule for this decision problem (either randomized or non-randomized), and find a prior distribution for θ for which this minimax rule is a Bayes rule.