STA 3000, Fall 2009 — Assignment #4

Due 15 January 2010. Worth 8% of the course grade.

This assignment is to be done by each student individually. You may discuss it in general terms with other students, but the work you hand in should be your own. In particular, you should not leave any discussion of this assignment with any written notes or other recordings, nor receive any written or other material from anyone else by other means such as email.

Consider the problem of estimating θ from two data points, $X_1, X_2 \overset{\text{IID}}{\sim} U(\theta, \theta+1)$. We will look at the following estimator:

$$\delta_0(x) \;=\; x_1 \;-\; 1/2$$

Consider first squared-error loss, for which $L(\theta, a) = (\theta - a)^2$, with a the real-valued estimate.

a) Find the risk function for δ_0.

b) Apply the Rao-Blackwell theorem to find an estimator δ_1 that should have risk at least as small as δ_0. As the sufficient statistic, use the order statistics, $X_{(1)}, X_{(2)}$

c) Find the risk function for δ_1.

d) Show that δ_1 is also the Pitman estimator for this problem, by correcting $\delta_0(x)$ by subtracting $E_{\theta=0}[\delta_0(X)|Y = x_1 - x_2]$, where $Y = X_1 - X_2$. Confirm that this is also what you get by finding the mean of the normalized likelihood function.

The Rao-Blackwell theorem applies only to convex loss functions. Consider instead the class, $\mathcal{M}$, of loss functions of the form $L(\theta, a) = f(|\theta - a|)$, with f being a monotonically non-decreasing function (ie, $f(d) \le f(d')$ if $d \le d'$).

e) For the specific case of this model, with δ_0 and δ_1 above, prove that for any loss function in the class $\mathcal{M}$, the risk for δ_1 is at least as small as the risk for δ_0.

Suppose that the model is instead that X_1 and X_2 are IID from a mixture distribution, with probability $9/10$ that X_i is exactly $\theta + 1/2$ and probability $1/10$ that X_i is drawn from the $U(\theta, \theta+1)$ distribution. In other words, $X_1, X_2 \overset{\text{IID}}{\sim} (9/10)\delta_{\theta+1/2} \;+\; (1/10)U(\theta, \theta+1)$, where δ_w is a point mass at w. We will look at the same estimator, δ_0, as above. If we use the order statistics as the sufficient statistic, the Rao-Blackwell theorem applied to δ_0 will give δ_1 as before.

f) Find a loss function in the class $\mathcal{M}$ for which the estimator δ_1 does not have risk at least as small as δ_0.