STA2503/MMF1941 Problem Set #4, due on December 1

You may submit your assignment to Hanna during her contact hours or leave it in my mailbox at SS6021.

In the following problems, $\{W(t)\}$ is always the standard Brownian motion.

1. Brownian Bridge For any real a and b, define the process $\{Y(t)\}$ as

$$Y(t) = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s} dW(s), \quad 0 \le t < 1.$$

The process $\{Y(t)\}$ is called the Brownian bridge from a to b.

(a) What is $\lim_{t\to 1} Y(t)$, under the 'in probability' convergence?

(b) Determine the stochastic differential equation to which $\{Y(t)\}$ is a solution.

2. Solve

(a)

$$dX(t) = X(t)dt + e^t dW(t);$$

(b)

$$dX(t) = -X(t)dt + e^{-t} dW(t);$$

3. Solve

$$dX(t) = -\frac{1}{1+t}X(t)dt + \frac{1}{1+t}dW(t),$$

for $X(0) = 0$ and $X(0) \neq 0$.

Hint: Consider $\frac{W(t)}{1+t}$.

4. Consider the SDE

$$dX(t) = rX(t)[\theta - X(t)]dt + \beta X(t)dW(t),$$

with r, θ, β and $X(0) = x$ being positive.

(a) Solve the equation using the integrating factor

$$Z(t) = e^{-\beta W(t) + \frac{1}{2}\beta^2 t}.$$

(b) Use the method in (a) to solve

$$dX(t) = \frac{1}{X(t)}dt + \beta X(t)dW(t),$$

and

$$dX(t) = X(t)^{\gamma} dt + \beta X(t) dW(t),$$

where $\gamma > 0$.

(c) If β is replaced by a deterministic function $\beta(t)$, find an integrating factor similar to that in (a) and use it solve the SDE

$$dX(t) = rX(t)[\theta - X(t)]dt + \beta(t)X(t)dW(t).$$

5. For the random variables $X_1(T) = [W(T)]^2$, $X_2(T) = \int_0^T W(s)ds$ and $X_3(T) = \sin W(T)$, find $\sigma_{X_i}(t), 0 \leq t \leq T$, defined in (5.77), for $i = 1, 2, 3$. What are the corresponding martingales in these cases.