The University of Toronto

ACT451/STA 2500 Loss Models Midterm Test

You must show your steps or no point will be awarded

Name____________________

ID # Solution.

1. The ground-up loss X has a gamma distribution with $\alpha = 4$ and $\theta = 10$. Let $Y = X^{-1}$.

a) **5 points** Determine the distribution function of Y explicitly.

$$F_Y(y) = P(Y \le y) = P(X^{-1} \le y) = P\left(X \ge \frac{1}{y}\right) = S_X\left(\frac{1}{y}\right).$$

Sol'n #1 Recall from lecture,

$X \sim \text{Erlang}(i, \theta)$ $\quad i = \alpha = 4,\ \theta = 10$

$$\Rightarrow S_X(x) = e^{-\frac{x}{\theta}}\left[\sum_{j=0}^{i-1} \frac{x^j}{\theta^j j!}\right]$$

$$\Rightarrow S_X\left(\tfrac{1}{y}\right) = e^{-\frac{1}{y\theta}}\left[\sum_{j=0}^{i-1} \frac{1}{(y\theta)^j j!}\right]$$

$$(\theta = 10) \quad = e^{-\frac{1}{10y}}\left[\sum_{j=0}^{3} \frac{1}{(10y)^j j!}\right] //$$

Sol'n #2

$$S_X\left(\tfrac{1}{y}\right) = 1 - F_X\left(\tfrac{1}{y}\right) = 1 - \Gamma\left(\alpha; \tfrac{1}{y\theta}\right)$$

$$\text{where } \Gamma\left(\alpha; \tfrac{1}{y\theta}\right) = \int_0^{1/y} \frac{u^{\alpha} e^{-u/\theta}}{\theta^{\alpha}\, \Gamma(\alpha)}\, du$$

$$\Rightarrow \Gamma\left(4; \tfrac{1}{10y}\right) = \int_0^{\frac{1}{y}} \frac{u^3 e^{-\frac{u}{10}}}{10^4 \cdot 3!}\, du$$

$$= \frac{1}{10^4\, 3!}\, e^{-\frac{u}{10}}\left(-u^3 \cdot 10 - 3u^2 \cdot 10^2 - 3\times 2\, u \cdot 10^3 - 3\times 2\times 1 \cdot 10^4\right)\Big|_0^{\frac{1}{y}}$$

$$= 1 - \underbrace{e^{-\frac{1}{10y}}\left\{\frac{1}{3!\,(10y)^3} + \frac{1}{2!\,(10y)^2} + \frac{1}{10y} + 1\right\}}_{S_X\left(\frac{1}{y}\right)} //.$$

b) **5 points** Determine the second raw moment $E(Y^2)$.

From table A.3.2.1 (Gamma Distn)

$$E(Y^2) = E(X^{-2})$$

$$= \theta^{-2}\, \Gamma(\alpha - 2) / \Gamma(\alpha)$$

$$= 10^{-2}\, \Gamma(4-2) / \Gamma(4)$$

$$= \frac{1}{600} //.$$

2. A portfolio of insurance policies consists of two types of policies. Losses from Type 1 policies have a gamma distribution with $\alpha = 4$ and $\theta = 10$. Losses from Type 2 policies have a gamma distribution with $\alpha = 2$ and $\theta = 10$. The policies are evenly divided between two types. A policy is chosen at random from the portfolio.

a) **5 points** Determine the variance of a loss from the chosen policy.

For two-point mixture with equal weights,

we have:

$$f(x) = \frac{1}{2}\left[f_1(x) + f_2(x) \right]$$

$$E(X^k) = \frac{1}{2}\left[E_1(X^k) + E_2(X^k) \right]$$

From Table A.3.2.1 For Gamma Distn, we have:

$$E(X) = \theta\alpha \qquad X \sim \text{Gamma}(\alpha, \theta).$$

$$E(X^2) = \theta^2(\alpha+1)\alpha$$

$$\Rightarrow \begin{cases} \text{Type 1:} & E_1(X) = 40, \quad E_1(X^2) = 10^2(5)(4) = 2000 \\ \text{Type 2:} & E_2(X) = 20, \quad E_2(X^2) = 10^2(3)(2) = 600 \end{cases}$$

Mixture: $E(X) = \frac{1}{2}(40+20) = 30$

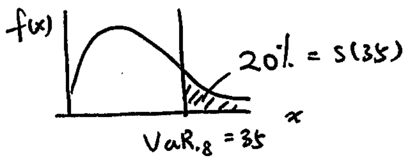
$$E(X^2) = \frac{1}{2}(2000 + 600) = 1300$$

$$\therefore \quad Var(X) = E(X^2) - E^2(X)$$

$$= 1300 - 30^2$$

$$= \underline{\underline{400}}. \quad //$$

b) **5 points** Suppose that the value at risk at confidence level 80% for the chosen policy is 35. Determine the CTE(80) for the policy.



$$CTE(80\%) = E(X-35 \mid X>35) + 35$$

$$= \frac{E(X-35)_+}{S(35)} + 35 \qquad (*)$$

$S(35) = 0.2$ since $VaR_{.8} = 35$

Now, to calc. $E(X-d)_+$ where $X \sim \text{Erlang}(i, \theta)$,
we can use the result derived from Lecture:

$$E(X-d)_+ = \int_d^{\infty} S(y)\,dy = \theta\left[\sum_{j=1}^{i-1}\sum_{k=0}^{j} \frac{d^k}{\theta^k k!}\right] e^{-\frac{d}{\theta}}$$

$\Rightarrow$ Type 1: $E_1(X-35)_+ = 10\left(\sum_{j=0}^{4-1}\sum_{k=0}^{j}\left(\frac{35}{10}\right)^k \cdot \frac{1}{k!}\right)e^{-\frac{35}{10}} = 10\left(4 + 3\times3.5 + 2\times\frac{3.5^2}{2} + \frac{3.5^3}{3!}\right)e^{-3.5}$

$= 10.23565$

Type 2: $E_2(X-35)_+ = 10\left(\sum_{j=0}^{2-1}\sum_{k=0}^{j}\left(\frac{35}{10}\right)^k \cdot \frac{1}{k!}\right)e^{-\frac{35}{10}} = 10\,(2\times1 + 3.5)\,e^{-3.5}$

$= 1.6608$

Mixture (equal weight):

$$E(X-35)_+ = \frac{1}{2}(10.23565 + 1.6608) = 5.94825$$

Continue from (*).

$$CTE(80\%) = \frac{5.94825}{0.2} + 35 = \underline{\underline{64.74}} \quad //$$

c) **5 points** Suppose that a policy limit of 45 is imposed on all the losses. Determine the expected cost per loss from the chosen policy.

$$E(X \wedge 45) = \underbrace{E(X)}_{=30 \text{ from (a)}} - E(X-45)_+$$

Type 1: $E_1(X-45)_+ = 10\left(4 + 3\times 4.5 + 2\times \frac{4.5^2}{2!} + \frac{4.5^3}{3!}\right)e^{-4.5}$

$= 5.88$

Type 2: $E_2(X-45)_+ = 10\,(2 + 4.5)e^{-4.5}$

$= 0.722$

Mixture: $E(X-45)_+ = \frac{1}{2}(5.88 + 0.722) = 3.30$

$\Rightarrow E(X \wedge 45) = 30 - 3.30 = \underline{26.70}$. //

d) **5 points** Instead of a policy limit, an ordinary deductible of 10 is imposed on all the losses. Determine the expected cost per loss from the chosen policy.

$$E(X-10)_+ = \frac{1}{2}\left[E_1(X-10)_+ + E_2(X-10)_+\right] = \underline{20.54} \text{ //}$$

$10\,(2+1)\,e^{-1}$

$10\left(4 + 3 + \frac{2}{2} + \frac{1}{3!}\right)e^{-1}$

3. 5 points The random variable Λ has a gamma distribution with parameters α and θ. The conditional loss random variable X, given $\Lambda = \lambda$, has an inverse exponential distribution with parameter λ. Determine the unconditional distribution of X.

Sol'n #1

$$F_X(x) = E_\Lambda\left[F_{X|\Lambda}(x \mid \Lambda = \lambda)\right]$$

$$= E_\Lambda\left(e^{-\lambda/x}\right)$$

$$= M_\Lambda\left(-\tfrac{1}{x}\right) \qquad \text{where } \Lambda \sim \text{Gamma}(\alpha, \theta)$$

$$= \frac{1}{1 - \theta\left(-\frac{1}{x}\right)^{\alpha}}$$

$$= \frac{x^\alpha}{(x+\theta)^\alpha} \qquad \therefore\ X \sim \text{Inv. Pareto}(\alpha, \theta).$$

Sol'n #2

$$F_X(x) = E_\Lambda\left[F_{X|\Lambda}(x \mid \Lambda = \lambda)\right]$$

$$= E_\Lambda\left(e^{-\lambda/x}\right)$$

$$= \int_0^\infty e^{-\frac{\lambda}{x}} \frac{\left(\frac{\lambda}{\theta}\right)^\alpha e^{-\lambda/\theta}}{\lambda\,\Gamma(\alpha)}\, d\lambda$$

$$= \frac{1}{\theta^\alpha\left(\frac{1}{\theta}+\frac{1}{x}\right)^\alpha} \underbrace{\int_0^\infty \frac{\frac{\lambda^\alpha}{\left[\left(\frac{1}{\theta}+\frac{1}{x}\right)^\alpha\right]^{-1}} e^{-\lambda/\left[\frac{1}{\theta}+\frac{1}{x}\right]^{-1}}}{\lambda\,\Gamma(\alpha)}\, d\lambda}_{=1 \text{ since this is pdf of Gamma}\left(\alpha+1,\ \left[\frac{1}{\theta}+\frac{1}{x}\right]^{-1}\right).}$$

$$= \frac{x^\alpha}{(x+\theta)^\alpha}. \qquad \therefore\ X \sim \text{Inv. Pareto}(\alpha, \theta).$$

4. The ground-up loss X for a particular policy has a lognormal distribution with $\mu = 2$ and $\sigma = 2$. If the issuer of the policy purchases a reinsurance policy such that the excess of the loss over 45 will be covered by the reinsurer.

a) **5 points** Determine the LER for this policy.

$$X \sim \text{lognormal } (\mu = 2, \ \sigma = 2).$$

$$LER = \frac{E(X \wedge 45)}{E(X)}$$

$$E(X) = e^{\mu + \frac{\sigma^2}{2}} = e^4$$

$$E(X \wedge 45) = e^{\mu + \frac{\sigma^2}{2}} \Phi\Big(\underbrace{\frac{\ln 45 - \mu - \sigma^2}{\sigma}}_{\frac{\ln 45 - 2 - 4}{2} = -1.10}\Big) + 45\Big(1 - \Phi\Big(\underbrace{\frac{\ln 45 - \mu}{\sigma}}_{\frac{\ln 45 - 2}{2} = 0.9}\Big)\Big).$$

$$= e^4 \, \Phi(-1.10) + 45(1 - \Phi(0.9))$$

$$= e^4 \times 0.1357 + 45 \times 0.1841$$

$$= 15.69347$$

$$\Rightarrow LER = \frac{15.69347}{e^4} = 0.2874$$

b) **5 points** Determine the variance of cost per loss to the ceding insurer.

$$V(X\wedge 45) = E((X\wedge 45)^2) - \underbrace{[E(X\wedge 45)]^2}_{15.69347 \text{ from (a)}}$$

$$E((X\wedge 45)^2) = e^{2\mu+\frac{2^2\sigma^2}{2}}\,\Phi\left(\frac{\ln 45-\mu-2\sigma^2}{\sigma}\right) + 45^2\underbrace{\left[1-\Phi\left(\frac{\ln 45-\mu}{\sigma}\right)\right]}_{0.1841 \text{ from (a)}}$$

$$= e^{12}\underbrace{\Phi(-3.10)}_{0.001} + 45^2[0.1841]$$

$$= 535.56$$

$$\Rightarrow V(X\wedge 45) = 535.56 - 15.69347^2 = 289.275$$

c) 5 points A ceded premium for the reinsurance policy is determined so that the probability that the payment by the reinsurer would be less than the premium is 0.9. Determine the premium.

$P(x - 45 < prem) = 0.9$ where $x \sim$ lognormal $(\mu = 2, \sigma = 2)$

$\Rightarrow \Phi\left(\frac{\ln(prem + 45) - 2}{2}\right) = 0.9$

$\Rightarrow prem = \exp\{1.282 \times 2 + 2\} - 45$

$= 50.97$ //