

Option pricing with the binomial model

As is shown earlier, the binomial model has the unique risk neutral probability measure Q such that

$$P_Q\{Y_t = h_1\} = q, \quad P_Q\{Y_t = -h_2\} = 1 - q,$$

where $q = \frac{1+r-d}{u-d}$, and the time-0 price of a European option with payoff $C(T)$, maturing at time T , is expressed as

$$\phi_C = E_Q \left\{ (1+r)^{-T} C(T) \right\}. \quad (1)$$

We now evaluate a European call option written on the stock $S(t)$ with a strike price of K , i.e. it has the payoff

$$C(T) = \max\{S(T) - K, 0\}.$$

Obviously, for any t , the stock price $S(t)$ under the risk-neutral probability measure is binomially distributed and

$$P_Q\{S(t) = S(0)u^s d^{t-s}\} = \binom{t}{s} q^s (1-q)^{t-s}, \quad s = 0, 1, \dots, t, \quad (2)$$

for $t = 1, 2, \dots, T$. Hence, the time-0 price of the call is given by

$$\begin{aligned} \phi_{call} &= (1+r)^{-T} E_Q(C) \\ &= (1+r)^{-T} \sum_{s=0}^T \max\{S(0)u^s d^{T-s} - K, 0\} \binom{T}{s} q^s (1-q)^{T-s} \\ &= (1+r)^{-T} \sum_{s \geq \frac{\log(K/S(0)) - T \log d}{\log(u/d)}} \{S(0)u^s d^{T-s} - K\} \binom{T}{s} q^s (1-q)^{T-s} \end{aligned}$$

$$\begin{aligned}
&= (1+r)^{-T}S(0) \sum_{s \geq \frac{\log(K/S(0)) - T \log d}{\log(u/d)}} \binom{T}{s} u^s d^{T-s} q^s (1-q)^{T-s} \\
&- (1+r)^{-T}K \sum_{s \geq \frac{\log(K/S(0)) - T \log d}{\log(u/d)}} \binom{T}{s} q^s (1-q)^{T-s}.
\end{aligned}$$

Let

$$\hat{q} = \frac{uq}{1+r}. \quad (3)$$

Then,

$$1 - \hat{q} = \frac{dq}{1+r}. \quad (4)$$

The price of the call can then be written as

$$\begin{aligned}
\phi_{call} &= S(0) \sum_{s \geq \frac{\log(K/S(0)) - T \log d}{\log(u/d)}} \binom{T}{s} \hat{q}^s (1 - \hat{q})^{T-s} \\
&- (1+r)^{-T}K \sum_{s \geq \frac{\log(K/S(0)) - T \log d}{\log(u/d)}} \binom{T}{s} q^s (1-q)^{T-s} \\
&= S(0) \sum_{s \leq \frac{T \log u + \log(S(0)/K)}{\log(u/d)}} \binom{\bar{T}}{s} (1 - \hat{q})^s \hat{q}^{T-s} \\
&- (1+r)^{-T}K \sum_{s \leq \frac{T \log u + \log(S(0)/K)}{\log(u/d)}} \binom{T}{s} (1-q)^s q^{T-s}. \quad (5)
\end{aligned}$$

Denote $x_0 = \frac{\bar{T} \log u + \log(S(0)/K)}{\log(u/d)}$ and $B(x; n, p)$ the distribution function of the binomial distribution with parameters n and p , i.e.

$$B(x; n, p) = \sum_{s \leq x} \binom{n}{s} p^s (1-p)^{n-s}. \quad (6)$$

We have

$$\phi_{call} = S(0)B(x_0; T, 1 - \hat{q}) - (1+r)^{-T}KB(x_0; T, 1 - q). \quad (7)$$

This formula is the well known option pricing formula of Ross, Cox and Rubinstein. The delta is $B(x_0; T, 1 - \hat{q})$. Other greeks can be calculated easily. It also reveals that to replicate a European call, the strategy is to form a portfolio long in stock and short in bond. Note that the first summation in the formula is the distribution function of the binomial distribution with parameter $1 - \hat{q}$ and the second summation is the distribution function of the binomial distribution with parameter $1 - q$. Hence its price can be evaluated quite easily.

To price a European put option, we can either use a approach similar to the above or the put-call parity as follows.

Let

$$P(T) = \max\{K - S(T), 0\}$$

be the payoff of a European put option with a strike price of K . It is easy to see that

$$\max\{S(T) - K, 0\} - \max\{K - S(T), 0\} = S(T) - K.$$

Hence if ϕ_{put} is the price of the put, we have

$$\phi_{call} - \phi_{put} = (1 + r)^{-T}[E_Q(S(T)) - K] = S(0) - (1 + r)^{-T}K. \quad (8)$$

This identity (8) is called the put-call parity.

We now consider the valuation of American options. The payoff structure of an American option is similar to its European counterpart. The difference is that an American option can be exercised at anytime before its maturity. For example, an American call option and an American put option written on a stock $S(t)$ for the period $[0, T]$ can be exercised before or at time T .

Their payoffs, if exercised at t , are $\max(S(t) - K, 0)$ and $\max(K - S(t), 0)$, respectively.

The valuation of American options is generally much more difficult than European options. There are no closed form solutions for American options. This is because the buyer of an American option holds the right to exercise at anytime and the valuation problem becomes how to find the optimal exercise time at which the expected discounted payoff for the buyer is maximized. Since a decision on whether to exercise should be based on the information up to date, an exercise time is a stopping time.

Let $g(S(t), t)$ be the payoff of an American option if it is exercised at time t . If the decision to exercise this option is based on a stopping time \mathcal{T} , then the price of this option is

$$E_Q\{(1+r)^{-\mathcal{T}}g(S(\mathcal{T}), \mathcal{T})\}. \quad (9)$$

One should assume that the buyer of the American option always wants to maximize the expected discounted payoff. Hence the price of this option is

$$\phi_g = \max_{\mathcal{T}} E_Q\{(1+r)^{-\mathcal{T}}g(S(\mathcal{T}), \mathcal{T})\}. \quad (10)$$

The maximization is taken over all stopping times over the period $[0, T]$. There is no put-call parity for American options since the optimal exercise time for a call is different from the optimal exercise time for the corresponding put.

It is impractical to examine each of these stopping times in (10) in order to find the optimal exercise time and the price of the option. However, under the discrete-time framework one is able to find the optimal exercise time and the option price through a backward recursive algorithm as follows.

We begin with the last time interval. For $t = T$, define a random variable $v(T - 1)$ on $(\Omega, \mathcal{F}_{T-1})$ as

$$v(T - 1) = \max \{(1 + r)^{-1} E_Q \{g(S(T), T) | \mathcal{F}_{T-1}\}, g(S(T - 1), T - 1)\}. \quad (11)$$

For $t = 1, \dots, T - 1$, define a random variable $v(t - 1)$ on $(\Omega, \mathcal{F}_{t-1})$ as

$$v(t - 1) = \max \{(1 + r)^{-1} E_Q \{v(t) | \mathcal{F}_{t-1}\}, g(S(t - 1), t - 1)\}. \quad (12)$$

The value $v(0)$ is then the price of this American option at time 0. Furthermore, $v(t)$ is the value of the option at time t . In other words, the value of an American option is calculated as the maximum of the expected discounted value of the same option at next trading date and the current payoff. The optimal exercise time of this option is

$$\mathcal{T}_g = \min\{t; g(S(t), t) > v(t)\}. \quad (13)$$

If the set is empty, we define $\mathcal{T}_g = T$.

The rationale behind this algorithm is quite simple. We choose $\mathcal{T}_0 = T$ as an initial exercise time which of course is not optimal. If at a node at time $T - 1$, say F_{T-1}^i ,

$$g(S(T - 1), T - 1) > (1 + r)^{-1} E_Q \{g(S(T), T) | F_{T-1}^i\},$$

we define

$$\mathcal{T}_1 = \begin{cases} T - 1, & \text{at } F_{T-1}^i \\ \mathcal{T}_0, & \text{otherwise.} \end{cases}$$

Thus \mathcal{T}_1 will yield a higher expected discounted payoff than \mathcal{T}_0 . The same argument applies to first the nodes at time $T - 1$ and then the nodes on

intermediate trading dates backwards. After we go through all the nodes we obtain the optimal exercise time and the value of the option.

The algorithm we have discussed above is quite flexible. It can apply to other types of options. For instance, we may use it to evaluate Bermudan options which allow their buyer to exercise the options during a given period of time before or at the expiration. In that case, we may use the algorithm for the exercise period and use an option pricing formula for European options for the no exercise period.

Finally, we show that there will never be an early exercise for an American call¹. Thus, the value of an American call is the same as that of the corresponding European call. To see this it is sufficient to show

$$\max\{S(t-1) - K, 0\} \leq (1+r)^{-1} E_Q\{\max\{S(t) - K, 0\} | \mathcal{F}_{t-1}\}. \quad (14)$$

The inequality (14) is in fact a direct application of the Jensen's inequality which states that for any random variable X and for any convex function $h(x)$, $h(E(X)) \leq E(h(X))$. Now choose $h(S) = \max\{S - K, 0\}$. We have

$$\begin{aligned} & (1+r)^{-1} E_Q\{\max\{S(t) - K, 0\} | \mathcal{F}_{t-1}\} \\ & \geq (1+r)^{-1} \max\{E_Q\{S(t) | \mathcal{F}_{t-1}\} - K, 0\} \\ & = \max\{S(t-1) - \frac{K}{1+r}, 0\} \geq \max\{S(t-1) - K, 0\}. \end{aligned}$$

¹We assume that the stock pays no dividends. Otherwise, the statement is not true.