probability measure under which a stochastic process is a martingale is called a *martingale measure*. In the following, we show how to find a martingale measure for a geometric random walk and extend the method to a more general case in the context of finance.

Let $\{S(t)\}$ be the price of a stock and assume that $\{S(t)\}$ follows a geometric random walk with drift μ and volatility σ. Thus, $S(t) = S(0)e^{X(t)}$, where $X(t) = Y_1 + Y_2 + \cdots + Y_t$, and $Y_t, t = 1, 2, \cdots$, are iid binomial random variables with

$$P\{Y_t = h_1\} = P\{Y_t = -h_2\} = \frac{1}{2}.$$

The values of h_1 and h_2 are given in (3.11). Further let r be the interest rate for each time interval $[t, t+1]$. Thus, $V(t) = (1+r)^{-t}S(t)$ represents the present value of the stock price process. If $r = 0$, $V(t) = S(t)$, a geometric random walk. We now illustrate how to change the probability measure to a martingale measure Q under which the present value process $V(t)$ is a martingale. Suppose that under Q

$$P_Q\{Y_t = h_1\} = q, \quad P_Q\{Y_t = -h_2\} = 1 - q,$$

where $P_Q(\cdot)$ denotes probability under the martingale measure Q. Then, from (3.49), $\{V(t)\}$ is a martingale if

$$E_Q\{V(t+1)/V(t)\ |V(t)\} = 1,$$

where E_Q denotes the expectation under the martingale measure Q. Since $V(t+1)/V(t) = (1+r)^{-1}e^{Y_{t+1}}$, the equation above implies

$$uq + d(1-q) = 1 + r, \tag{3.53}$$

where $u = e^{h_1}$ and $d = e^{-h_2}$. Solving (3.53), we obtain

$$q = \frac{1 + r - d}{u - d}.$$

Note that (3.5) is the same as the risk neutral probability for the one-period model using a delta hedging strategy. This martingale measure is often called the risk neutral measure for the binomial model in option pricing and the time-0 price of a European option with payoff $C(T)$, maturing at time T, is expressed as

$$\phi_C = E_Q\left\{(1 + r)^{-T} C(T)\right\}. \tag{3.54}$$

Consider now a more general case where $Y_t,\ t = 1, 2, \cdots$, are independent but arbitrary random variables.

Assume that the moment generating function $M_{Y_t}(z)$ of Y_t exists for all t. Define, for any real value α_t,

$$Q(t) = e^{\alpha_t Y_t} / M_{Y_t}(\alpha_t).$$

Then, $Q(t), t = 1, 2, \cdots$, are independent, and $E\{Q(t)\} = 1$. Further define

$$Z(t) = Q(1)Q(2) \cdots Q(t). \tag{3.55}$$

$\{Z(t)\}$ is a strictly positive martingale under the old measure since

$$E\{Z(t + 1)/Z(t)|Z(t)\} = E\{Q(t + 1)\} = 1.$$

Now for any event A at time t, define the probability of A as

$$P_Q(A) = E\{\mathbf{I}_A Z(t)\}. \tag{3.56}$$