Appendix C

Common Distributions

We record here the most commonly used distributions in probability and statistics as well as some of their basic characteristics.

C.1 Discrete Distributions

1. *Bernoulli*(θ), $\theta \in [0, 1]$ (same as Binomial$(1, \theta)$).
probability function: $p(x) = \theta^x (1-\theta)^{1-x}$ for $x = 0, 1$.
mean: θ.
variance: $\theta(1-\theta)$.
moment-generating function: $m(t) = (1 - \theta + \theta e^t)$ for $t \in R^1$.

2. *Binomial*(n, θ), $n > 0$ an integer, $\theta \in [0, 1]$.
probability function: $p(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$ for $x = 0, 1, \ldots, n$.
mean: $n\theta$.
variance: $n\theta(1-\theta)$.
moment-generating function: $m(t) = (1 - \theta + \theta e^t)^n$ for $t \in R^1$.

3. *Geometric*(θ), $\theta \in (0, 1]$ (same as Negative-Binomial$(1, \theta)$).
probability function: $p(x) = (1-\theta)^x\theta$ for $x = 0, 1, 2, \ldots$.
mean: $(1-\theta)/\theta$.
variance: $(1-\theta)/\theta^2$.
moment-generating function: $m(t) = \theta(1 - (1-\theta)\, e^t)^{-1}$ for $t < -\ln(1-\theta)$.

4. *Hypergeometric*(N, M, n), $M \le N$, $n \le N$ all positive integers.
probability function:

$$p(x) = \binom{M}{x}\binom{N-M}{n-x}\Big/\binom{N}{n} \text{ for } \max(0,\ n + M - N) \le x \le \min(n, M).$$

mean: $n\frac{M}{N}$.
variance: $n\frac{M}{N}\left(1 - \frac{M}{N}\right)\frac{N-n}{N-1}$.

5. *Multinomial*$(n, \theta_1, \ldots, \theta_k)$, $n > 0$ an integer, each $\theta_i \in [0, 1]$, $\theta_1 + \cdots + \theta_k = 1$.

probability function:

$$p(x_1, \dots, x_k) \;=\; \binom{n}{x_1 \dots x_k} \theta_1^{x_1} \cdots \theta_k^{x_k} \text{ where each } x_i \in \{0, 1, \dots, n\} \text{ and } x_1 + \cdots + x_k = n.$$

mean: $E(X_i) = n\theta_i$.
variance: $\mathrm{Var}(X_i) = n\theta_i(1 - \theta_i)$.
covariance: $\mathrm{Cov}(X_i, X_j) = -n\theta_i\theta_j$ when $i \neq j$.

6. *Negative-Binomial*(r, θ), $r > 0$ an integer, $\theta \in (0, 1]$.
probability function: $p(x) = \binom{r-1+x}{x}\theta^r(1 - \theta)^x$ for $x = 0, 1, 2, 3, \dots$.
mean: $r(1 - \theta)/\theta$.
variance: $r(1 - \theta)/\theta^2$.
moment-generating function: $m(t) = \theta^r(1 - (1 - \theta)\, e^t)^{-r}$ for $t < -\ln(1 - \theta)$.

7. *Poisson*(λ), $\lambda > 0$.
probability function: $p(x) = \frac{\lambda^x}{x!}e^{-\lambda}$ for $x = 0, 1, 2, 3, \dots$.
mean: λ.
variance: λ.
moment-generating function: $m(t) = \exp\{\lambda(e^t - 1)\}$ for $t \in R^1$.

C.2 Absolutely Continuous Distributions

1. *Beta*(a, b), $a > 0$, $b > 0$ (same as Dirichlet(a, b)).
density function: $f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1 - x)^{b-1}$ for $x \in (0, 1)$.
mean: $a/(a + b)$.
variance: $ab/(a + b + 1)(a + b)^2$.

2. *Bivariate Normal*$(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ for $\mu_1, \mu_2 \in R^1$, $\sigma_1^2, \sigma_2^2 > 0$, $\rho \in [-1, 1]$.
density function:

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp\left\{-\frac{1}{2(1 - \rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]\right\}$$

for $x_1 \in R^1$, $x_2 \in R^1$.

mean: $E(X_i) = \mu_i$.
variance: $\mathrm{Var}(X_i) = \sigma_i^2$.
covariance: $\mathrm{Cov}(X_1, X_2) = \rho\sigma_1\sigma_2$.

3. *Chi-squared*(α) or $\chi^2(\alpha)$, $\alpha > 0$ (same as Gamma$(\alpha/2, 1/2)$).
density function: $f(x) = 2^{-\alpha/2}(\Gamma(\alpha/2))^{-1}x^{(\alpha/2)-1}e^{-x/2}$ for $x > 0$.
mean: α.
variance: 2α.

moment-generating function: $m(t) = (1 - 2t)^{-\alpha/2}$ for $t < 1/2$.

4. *Dirichlet*$(\alpha_1, \ldots, \alpha_{k+1})$, $\alpha_i > 0$ for each i.
density function:

$$f_{X_1,\ldots,X_k}(x_1, \ldots, x_k) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_{k+1})}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_{k+1})} x_1^{\alpha_1 - 1} \cdots x_k^{\alpha_k - 1} (1 - x_1 - \cdots - x_k)^{\alpha_{k+1} - 1}$$

for $x_i \geq 0$, $i = 1, \ldots, k$ and $0 \leq x_1 + \cdots + x_k \leq 1$.

mean:

$$E(X_i) = \frac{\alpha_i}{\alpha_1 + \cdots + \alpha_{k+1}}.$$

variance:

$$\mathrm{Var}(X_i) = \frac{\alpha_i(\alpha_1 + \cdots + \alpha_{k+1} - \alpha_i)}{(\alpha_1 + \cdots + \alpha_{k+1})^2(1 + \alpha_1 + \cdots + \alpha_{k+1})}.$$

covariance when $i \neq j$:

$$\mathrm{Cov}(X_i, X_j) = \frac{-\alpha_i \alpha_j}{(\alpha_1 + \cdots + \alpha_{k+1})^2(1 + \alpha_1 + \cdots + \alpha_{k+1})}.$$

5. *Exponential*(λ), $\lambda > 0$ (same as Gamma$(1, \lambda)$).
density function: $f(x) = \lambda e^{-\lambda x}$ for $x > 0$.
mean: λ^{-1}.
variance: λ^{-2}.
moment-generating function: $m(t) = \lambda(\lambda - t)^{-1}$ for $t < \lambda$.

Note that some books and software packages instead replace λ by $1/\lambda$ in the definition of the Exponential(λ) distribution — always check this when using another book or when using software to generate from this distribution.

6. $F(\alpha, \beta)$, $\alpha > 0$, $\beta > 0$.
density function:

$$f(x) = \frac{\Gamma\left(\frac{\alpha+\beta}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)} \left(\frac{\alpha}{\beta}x\right)^{\alpha/2-1} \left(1 + \frac{\alpha}{\beta}x\right)^{-(\alpha+\beta)/2} \frac{\alpha}{\beta}$$

for $x > 0$.

mean: $\beta/(\beta - 2)$ when $\beta > 2$.
variance: $2\beta^2(\alpha + \beta - 2)/\alpha(\beta - 2)^2(\beta - 4)$ when $\beta > 4$.

7. *Gamma*(α, λ), $\alpha > 0$, $\lambda > 0$.
density function: $f(x) = \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x}$ for $x > 0$.
mean: α/λ.
variance: α/λ^2.
moment-generating function: $m(t) = \lambda^\alpha(\lambda - t)^{-\alpha}$ for $t < \lambda$.

Note that some books and software packages instead replace λ by $1/\lambda$ in the definition of the Gamma(α, λ) distribution — always check this when using another book or when using software to generate from this distribution.

8. Lognormal or $\log N(\mu, \sigma^2)$, $\mu \in R^1$, $\sigma^2 > 0$.

density function: $f(x) = (2\pi\sigma^2)^{-1/2} x^{-1} \exp\left(-\frac{1}{2\sigma^2}(\ln x - \mu)^2\right)$ for $x > 0$.
mean: $\exp(\mu + \sigma^2/2)$.
variance: $\exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1)$.

9. $N(\mu, \sigma^2)$, $\mu \in R^1$, $\sigma^2 > 0$.

density function: $f(x) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right)$ for $x \in R^1$.
mean: μ.
variance: σ^2.
moment-generating function: $m(t) = \exp(\mu t + \sigma^2 t^2/2)$ for $t \in R^1$.

10. *Student*(α) or $t(\alpha)$, $\alpha > 0$ ($\alpha = 1$ gives the Cauchy distribution).

density function:

$$f(x) = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} \left(1 + \frac{x^2}{\alpha}\right)^{-(\alpha+1)/2} \frac{1}{\sqrt{\alpha}}$$

for $x \in R^1$.

mean: 0 when $\alpha > 1$.
variance: $\alpha/(\alpha - 2)$ when $\alpha > 2$.

11. *Uniform*$[L, R]$, $R > L$.

density function: $f(x) = 1/(R - L)$ for $L < x < R$.
mean: $(L + R)/2$.
variance: $(R - L)^2/12$.
moment-generating function: $m(t) = (e^{Rt} - e^{Lt})/t(R - L)$.