Hence,

$$
\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}((m/n)uv,\, v)J^{-1}((m/n)uv,\, v) \\
&= \frac{\left(\frac{m}{n}uv\right)^{(m/2)-1} e^{-(m/n)(uv/2)} v^{(n/2)-1} e^{-(v/2)}}{2^{m/2}\Gamma\left(\frac{m}{2}\right) 2^{n/2}\Gamma\left(\frac{n}{2}\right)} \frac{m}{n} v \\
&= \frac{1}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}u\right)^{(m/2)-1} \frac{m}{n} \frac{1}{2^{(m+n)/2}} v^{(m+n)/2-1} e^{-(v/2)(1+mu/n)}
\end{aligned}
$$

for $u, v > 0$ (with $f_{U,V}(u,v) = 0$ for $u < 0$ or $v < 0$).

Finally, we compute the marginal density of U as

$$
\begin{aligned}
&f_U(u) \\
&= \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv \\
&= \frac{1}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}u\right)^{(m/2)-1} \frac{m}{n} \frac{1}{2^{(m+n)/2}} \int_0^{\infty} v^{(m+n)/2-1} e^{-(v/2)(1+mu/n)}\, dv \\
&= \frac{1}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}u\right)^{(m/2)-1} \left(1 + \frac{m}{n}u\right)^{-(n+m)/2} \frac{m}{n} \int_0^{\infty} w^{(m+n)/2-1} e^{-w}\, dw \\
&= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}u\right)^{(m/2)-1} \left(1 + \frac{m}{n}u\right)^{-(n+m)/2} \frac{m}{n},
\end{aligned}
$$

where we have used the substitution $w = (1 + mu/n)\, v/2$ to get the third equality, and the final result follows from the definition of the gamma function. ∎