Exercises on Decision Theory

1. Suppose $\mathcal{X} = \{1, 2, 3\}, \Theta = \{a, b, c\}$ and the f_θ are given by the following table.

$\theta \backslash x$	1	2	3
a	1/3	1/3	1/3
b	1/3	2/3	0
c	0	1/3	2/3

Suppose $\Psi(\theta) = I_{\{a\}}(\theta)$ and the loss function is 0-1 loss, namely, $Loss(\theta, \psi) = I_{\{\Psi(\theta)\}}(\psi)$.
(a) What is $\Psi(\Theta)$? Suppose $\delta \in \mathcal{D}(I^{dec})$ is given by $\delta(1, \{1\}) = 1, \delta(2, \{1\}) = 1/2, \delta(3, \{1\}) = 1/3$. Determine $R(\theta, \delta)$ for each value of θ.
(b) Suppose $d : \mathcal{X} \to \Psi(\Theta)$ is given by $d(1) = 1, d(2) = 0, d(3) = 0$. Is d the MLE estimate determined by profiling? What is $R(\theta, d)$ for each θ?
(c) Which is preferred between d and δ?
(d) Determine a mss T and then determine δ_T and d_T.
(e) Determine whether or not d is an unbiased estimator for this problem.

2. Suppose $\mathcal{X} = \{1, 2, 3, 4\}, \Theta = \{a, b\}$ and the f_θ are given by the following table.

$\theta \backslash x$	1	2	3	4
a	1/4	1/4	0	1/2
b	1/2	0	1/4	1/4

Suppose $\Psi(\theta) = \theta$ and the loss function is 0-1 loss, namely, $Loss(\theta, \psi) = I_{\{\Psi(\theta)\}}(\psi)$.
(a) Calculate the risk function for the nonandomized decision function given by $d(1) = d(2) = d(3) = a, d(4) = b$.
(b) Show d is not admissible?
(c) Suppose we add the prior given by $\pi(a) = 1/4, \pi(b) = 3/4$. Determine the prior risk of d.
(d) Obtain a Bayes rule based on the prior in (c). Is the Bayes rule admissible?

3. Suppose $x \sim$ Bernoulli(θ) where $\theta \in \Theta = \{1/2, 1\}$. After observing x we want to estimate θ when using 0-1 loss. Determine the minimax rule.

5. Suppose $\mathcal{X} = \{1, 2, 3\}, \Theta = \{a, b, c\}$ and the f_θ are given by the following table.

$\theta \backslash x$	1	2	3
a	1/6	1/3	1/2
b	1/2	1/6	1/3
c	1/3	1/2	1/6

(a) What is the largest group G of transformations on $\mathcal{X}$ that leave the model invariant.
(b) Is this a group model?
    Justify your answer.
(c) Suppose interest is in $\Psi(\theta) = \theta$. Show Ψ is equivariant. Is $\Psi(\theta) = I_{\{a,b\}}(\theta)$ is equivariant?
(d) Suppose $\Psi(\theta) = \theta$ and $Loss(\theta, \psi) = I_{\{\psi\}}(\theta)$. Show that the loss function is invariant under G
(e) Show that the MLE $\hat{\theta}(x)$ is equivaraint and is optimal among all equivariant estimators d for the problem in (d).
(f) Show that the MLE is optimal invariant among all invariant decision functions δ for the problem in (d).