Solutions 2.

Ex III.4.1

(i) For $x \in \mathbb{N}_0$, $f_\lambda(x) = \lambda^x e^{-\lambda}/x!$

$= \exp\{(\log\lambda)x - \lambda\}/x!$ so $\eta(\lambda) = \log\lambda$, $t(x) = x$
$A(\lambda) = \lambda$, $h(x) = 1/x!$

(ii) For $x \in [0,\infty)$ $\text{gamma}(\alpha,\lambda,x) = \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$

$= \exp\{(-\lambda)x + \alpha\log\lambda\} x^{\alpha-1}/\Gamma(\alpha)$ so

$\eta(\lambda) = -\lambda$, $t(x) = x$, $A(\lambda) = -\alpha\log\lambda$, $h(x) = x^{\alpha-1}/\Gamma(\alpha)$

(iii) For $x = (x_1,\dots,x_n)$ s.t. $0 \le x_i \le n$, $x_1 + \dots + x_n = n$

$$\binom{n}{x_1 \dots x_k} p_1^{x_1} \cdots p_k^{x_k} = \exp\{\sum_{i=1}^{k} p_i x_i\} \binom{n}{x_1 \dots x_k}$$

so $\eta(p_1,\dots,p_k) = (p_1,\dots,p_k)$ (not affinely independent)
$t(x_1,\dots,x_k) = (x_1,\dots,x_k)$, $A(p_1,\dots,p_k) = 0$
$h(x_1,\dots,x_k) = \binom{n}{x_1\, x_2 \dots x_k}$

(iv) For $\underline{x} \in \mathbb{R}^2$ $f_{(\underline{\mu},\Sigma)}(\underline{x}) = (2\pi)^{-1} |\det\Sigma|^{-1/2} \exp\{-(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})/2\}$

$= |\det\Sigma|^{-\frac{1}{2}} \exp\{-\frac{1}{2}\underline{x}'\Sigma^{-1}\underline{x} + \underline{x}'\Sigma^{-1}\underline{\mu} - \frac{1}{2}\underline{\mu}'\Sigma\underline{\mu}\}$. Since

$$\Sigma^{-1} = \frac{1}{\det\Sigma}\begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}$$ then

$$\underline{x}'\Sigma^{-1}\underline{x} = \frac{1}{\det\Sigma}(\sigma_{22}x_1^2 - 2\sigma_{12}x_1x_2 + \sigma_{11}x_2^2)$$ and

$$\underline{x}' \Sigma^{-1} \underline{\mu} = \frac{1}{\det \Sigma} (x_1, x_2) \begin{pmatrix} \sigma_{22}\mu_1 - \sigma_{12}\mu_2 \\ -\sigma_{12}\mu_1 + \sigma_{11}\mu_2 \end{pmatrix}$$

$$= \frac{1}{\det \Sigma} \left((\sigma_{22}\mu_1 - \sigma_{12}\mu_2) x_1 + (-\sigma_{12}\mu_1 + \sigma_{11}\mu_2) x_2 \right)$$

Therefore,

$$f_{\underline{\mu},\Sigma}(\underline{x}) = \exp\Big\{ -\frac{1}{\det \Sigma} \underbrace{\left(\frac{\sigma_{22}}{2}, -\sigma_{12}, \frac{\sigma_{11}}{2}, \sigma_{22}\mu_1 - \sigma_{12}\mu_2, -\sigma_{12}\mu_1 + \sigma_{11}\mu_2 \right)}_{\underline{\eta}(\underline{\mu},\Sigma)} \underbrace{\begin{pmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \\ x_1 \\ x_2 \end{pmatrix}}_{\underline{t}(\underline{x})}$$

$$\underbrace{- \frac{1}{2}\log \det \Sigma}_{A(\underline{\mu},\Sigma)} \Big\} \underbrace{\frac{1}{2\pi}}_{h(\underline{x})}$$

and the model has exponential form.

$\boxed{\text{Ex III.4.2}}$ Suppose $\exists\ a, b_1, \ldots, b_k$ s.t. $a + b_1\eta_1(\theta) + \cdots + b_k\eta_k(\theta) = 0$ $\forall \theta \in \Theta$. In particular this holds $\forall \theta \in B_\delta(\theta_0)$. But $B_\delta(\theta_0)$ is an open subset of $\mathbb{R}^k$ and no subset of a hyperplane is open in $\mathbb{R}^k$. Therefore, $\eta_1, \ldots, \eta_k$ must be affinely independent.

Note - this implies all the $\underline{t}$ fns in Ex III.4.1 are mss

Ex III.6.1

Clearly the invariance relation (parameter space) is

(i) reflexive: $\theta = \Psi(\theta)$ where Ψ = identity is 1-1

(ii) symmetric: if $\psi = \Psi(\theta)$ with Ψ 1-1, then $\theta = \Psi^{-1}(\psi)$ and Ψ^{-1} is 1-1

(iii) transitive: if $\psi = \Psi(\theta)$, $\eta = \Lambda(\psi)$ where Ψ and Λ are 1-1 then $\eta = \Lambda \circ \Psi(\theta)$ and $\Lambda \circ \Psi$ is 1-1

Ex III.6.2

When comparing inference bases we assume the parameter spaces are the same (perhaps after reparameterization).

(i) reflexive: if $I = (\{f_\theta : \theta \in \Theta\}, x)$ then

$L(\theta \mid x) = 1 \cdot L(\theta \mid x)\ \forall\, \theta \in \Theta$ so $(I, I) \in \mathcal{L}$

(ii) symmetric: if $I_i = \{ f_{i\theta} : \theta \in \Theta \}, x_i\}$ for $i = 1, 2$

and $(I_1, I_2) \in \mathcal{L}$ then $\exists\, c > 0$ s.t. $L(\theta \mid x_1) = c L(\theta \mid x_2)$

so $L(\theta \mid x_2) = \frac{1}{c} L(\theta \mid x_1)$ which implies $(I_2, I_1) \in \mathcal{L}$.

(iii) transitive: if $I_i = (\{ f_{i\theta} : \theta \in \Theta \}, x_i)$ and

$(I_1, I_2), (I_2, I_3) \in \mathcal{L}$ then $\exists\, c_1, c_2 > 0$ s.t.

$L(\theta \mid x_1) = c_1 L(\theta \mid x_2)$, $L(\theta \mid x_2) = c_2 L(\theta \mid x_3)$

so $L(\theta \mid x_1) = c_1 c_2 L(\theta \mid x_3)$ and $(I_1, I_3) \in \mathcal{L}$.

Ex III.6.3

Suppose we have inference bases $I_i = (\{f_{i\theta} : \theta \in \Omega\}, x_i)$ for $i = 1, 2, 3$ with mss's T_1, T_2, T_3 and models $\mathcal{M}_{T_1}, \mathcal{M}_{T_2}, \mathcal{M}_{T_3}$.

(i) reflexive : $(I_i, I_i) \in S$ because I_i, I_i have the same mss

(ii) symmetric : Suppose $(I_1, I_2) \in S$. Then $\exists$ 1-1 fn h s.t. $T_2 = h(T_1)$, $T_2(x_2) = h(T_1(x_1))$, $\mathcal{M}_{T_2} = \mathcal{M}_{h(T_1)}$ But then $T_1 = h^{-1}(T_2)$, $T_1(x_1) = h^{-1}(T_2(x_2))$ and $\mathcal{M}_{T_1} = \mathcal{M}_{h^{-1}(T_2)}$. Therefore using the invariance principle (sample space) we have that $(I_2, I_1) \in S$.

(iii) transitive : Suppose $(I_1, I_2), (I_2, I_3) \in S$. Then $\exists$ h, k 1-1 s.t. $T_3 = k(T_2)$, $T_2 = h(T_1)$ with $T_3(x_3) = k(T_2(x_2))$, $T_2(x_2) = h(T_1(x_1))$ and $\mathcal{M}_{T_3} = \mathcal{M}_{k(T_2)}$, $\mathcal{M}_{T_2} = \mathcal{M}_{h(T_1)}$ and so $T_3(x_3) = k \circ h(T_1(x_1))$, $\mathcal{M}_{T_3} = \mathcal{M}_{k \circ h(T_1)}$ and this implies $(I_1, I_3) \in S$.

$\boxed{\text{Ex III.6.4}}$

The model in I_1 should have been $\{\sum_{i=1}^{n} \text{Bernoulli}(\theta) : \theta \in (0,1)\}$ i.e. a sample of n iid values from a Bernoulli(θ). The mss for this model is $\sum_{i=1}^{n} x_i = \#$ of 1's $\sim$ binomial(n,θ). The model for I_2 is $\{$Negative binomial(k,θ) : $\theta \in (0,1)\}$ with mss the response x_2. So the mss's for the two models have quite different distributions.

$\boxed{\text{Ex III.6.5}}$

Suppose we have a class of equivalence relations $\{\mathcal{E}_\alpha : \alpha \in A\}$ each containing the relation R. Let $\mathcal{E} = \bigcap_{\alpha \in A} \mathcal{E}_\alpha$. Clearly $R \subseteq \mathcal{E}$.

Now R is a reflexive relation on a set D so $(d,d) \in R$ $\forall d \in D$ which implies $(d,d) \in \mathcal{E}_\alpha$ $\forall \alpha$ which implies $(d,d) \in \mathcal{E}$ and $\mathcal{E}$ is reflexive.

Now if $(d_1,d_2) \in R$ then $(d_1,d_2),(d_2,d_1) \in \mathcal{E}_\alpha$ $\forall \alpha$ because $\mathcal{E}_\alpha$ is an eq. rel containing R. Therefore

$(d_1,d_2),(d_2,d_1)\in\mathcal{E}$ and $\mathcal{E}$ is symmetric. Finally if $(d_1,d_2),(d_2,d_3)\in R$, then $(d_1,d_2),(d_2,d_3)\in\mathcal{E}_\alpha$ $\forall\alpha$ and $(d_1,d_3)\in\mathcal{E}_\alpha$ $\forall\alpha$ because $\mathcal{E}_\alpha$ is an eq. rel containing R. This implies $(d_1,d_2),(d_2,d_3),(d_1,d_3)$ $\in\mathcal{E}$ and $\mathcal{E}$ is transitive.

$\boxed{\text{Ex III.6.6}}$

It is clear that $h(i,x)=i$ is ancillary because $P_\theta(h(i,x)=i)=1/2$ $\forall\theta\in\Theta$ which is independent of θ. Also the conditional distribution of $(i,x)\mid T(i,x)$ is degenerate at (i,x) when $x\notin\{x_1,x_2\}$ and when $x\in\{x_1,x_2\}$ the conditional distribution is

$$\tfrac{1}{2}f_{M_i\theta}(x_i)\Big/\left(\tfrac{1}{2}f_{M_1\theta}(x_1)+\tfrac{1}{2}f_{M_2\theta}(x_2)\right)\quad\text{when } x=x_i$$

using $f_{M_1\theta}(x_1)=c\,f_{M_2\theta}(x_2)$

$$=\begin{cases}\tfrac{1}{2}f_{M_1\theta}(x_1)\Big/\left(\tfrac{1}{2}f_{M_1\theta}(x_1)+\tfrac{1}{2}c f_{M_2\theta}(x_2)\right) & \text{when } i=1\\ \tfrac{1}{2}f_{M_2\theta}(x_2)\Big/\left(\tfrac{1}{c}\tfrac{1}{2}f_{M_1\theta}(x_2)+\tfrac{1}{2}f_{M_2\theta}(x_2)\right) & \text{when } i=2\end{cases}$$

$$=\begin{cases}1/(1+c) & \text{when } i=1\\ (1/c+1) & \text{when } i=2.\end{cases}$$

In all cases the conditional distribution given $T(\hat{\imath}, x)$ does not depend on θ and so T is a sufficient statistic (Theorem III.3.2).

$\boxed{\text{Ex III.7.1}}$

Let $U_n = 2(1 - \Phi(|\sqrt{n}(\bar{x} - \mu_0)/\sigma_0|))$.

When $x_1, \ldots, x_n \overset{iid}{\sim} N(\mu, \sigma_0^2)$ then $\bar{x} \sim N(\mu, \sigma_0/\sqrt{n})$

so $Z = \sqrt{n}(\bar{x} - \mu_0)/\sigma_0 \sim N(\sqrt{n}(\mu - \mu_0)/\sigma_0, \sigma_0^2/n)$

When $\mu \neq \mu_0$ then for $u \in (0,1)$

$P(U_n \le u) = P(2(1 - \Phi(|Z|)) \le u)$

$= P(\Phi(|Z|) \geq 1 - u/2) = P(|Z| \geq \Phi^{-1}(1 - u/2))$

$= P(Z \le -\Phi^{-1}(1-u/2)) + P(Z \geq \Phi^{-1}(1-u/2))$

$= P(\sqrt{n}(\bar{x} - \mu)/\sigma_0 \le \sqrt{n}(\mu_0 - \mu)/\sqrt{n} - \Phi^{-1}(1-u/2))$

$\quad + P(\sqrt{n}(\bar{x} - \mu)/\sigma_0 \geq \sqrt{n}(\mu_0 - \mu)/\sqrt{n} + \Phi^{-1}(1-u/2))$

$= \Phi(\sqrt{n}(\mu_0 - \mu)/\sigma_0 - \Phi^{-1}(1-u/2)) + 1 - \Phi(\sqrt{n}(\mu_0 - \mu)/\sigma_0 + \Phi^{-1}(1-u/2))$

$\to 1$ when $\mu_0 > \mu$ $\left(\frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} \to \infty\right)$

$\to 1$ when $\mu_0 < \mu$ $\left(\frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} \to -\infty\right)$

$= \Phi(-\Phi^{-1}(1-u/2)) + 1 - \Phi(\Phi^{-1}(1-u/2))$ when $\mu = \mu_0$

$= 1 - \Phi(\Phi^{-1}(1-u/2)) + 1 - (1-u/2) = u$ when $\mu = \mu_0$

This proves that $U_n \xrightarrow{d} 0$ when $\mu \neq \mu_0$ since the limit holds for every $u \in (0,1)$ and that $U_n \sim U(0,1)$ when $\mu = \mu_0$ for every n.

$\boxed{\text{Ex. III.7.2}}$

When $\mu = \mu_0 \pm \delta$ we need to calculate

$$P(2(1-\Phi(|\sqrt{n}(\bar{x}-\mu_0)/\sigma_0|)) \le \alpha)$$

where $\bar{x} \sim N(\mu_0 \pm \delta, \sigma_0^2/n)$

$$= P(\Phi(|\sqrt{n}(\bar{x}-\mu_0)/\sigma_0|) \ge 1-\alpha/2)$$

$$= P(\sqrt{n}(\bar{x}-\mu_0)/\sigma_0 \le -\Phi^{-1}(1-\alpha/2)) + 1 - P(\sqrt{n}(\bar{x}-\mu_0)/\sigma_0 \le \Phi^{-1}(1-\alpha/2))$$

$$= P(\sqrt{n}(\bar{x}-\mu_0 \mp \delta)/\sigma_0 \le \mp\sqrt{n}\delta/\sigma_0 - \Phi^{-1}(1-\alpha/2)) + 1 - P(\sqrt{n}(\bar{x}-\mu_0 \mp \delta)/\sigma_0 \le \mp\sqrt{n}\delta/\sigma_0 + \Phi^{-1}(1-\alpha/2))$$

$$= \Phi(\mp\sqrt{n}\delta/\sigma_0 - \Phi^{-1}(1-\alpha/2)) + 1 - \Phi(\mp\sqrt{n}\delta/\sigma_0 + \Phi^{-1}(1-\alpha/2))$$

and we calculate this for $+\delta$ and $-\delta$ and take the maximum as the power

Note that since $\delta > 0$ the power $\to 1$ as $n \to \infty$ in both cases.