Solutions to Exercises - Lectures 10 and 11

II.6.1 Since $p \in (0,1]$ then $p_Y(i) = (1-p)^{i-1} p \geq 0$ for every $i \in \{1, 2, \ldots\}$. Further $\sum_{i=1}^{\infty} p_Y(i) = p \sum_{i=1}^{\infty} (1-p)^{i-1} = p \sum_{i=0}^{\infty} (1-p)^i$ which because it is a geometric series $= p \frac{1}{1-(1-p)} = p/p = 1$. Therefore p_Y is a valid probability function and so defines a probability distribution.

II.7.1 Since $X^{-1}\phi = \phi$ we have $\phi \in \mathcal{A}_X$. If $A_1, A_2, \ldots \in \mathcal{A}_X$ then $\exists\, B_1, B_2, \ldots \in \mathcal{B}^1$ s.t. $A_i = X^{-1}B_i$ which implies $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} X^{-1}B_i = X^{-1} \bigcup_{i=1}^{\infty} B_i$ and since $\bigcup_{i=1}^{\infty} B_i \in \mathcal{B}^1$ this implies $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_X$. If $A \in \mathcal{A}_X$ then $\exists\, B \in \mathcal{B}^1$ s.t. $A = X^{-1}B$ which implies $A^c = (X^{-1}B)^c = X^{-1}B^c$ which implies $A^c \in \mathcal{A}_X$ since $B^c \in \mathcal{B}^1$. Therefore $\mathcal{A}_X$ is a σ-algebra. Since $\mathcal{A}_X \subseteq \mathcal{A}$ it is a sub σ-algebra of $\mathcal{A}$.

II.7.2 (i) Suppose $\{X_\lambda : \lambda \in \Lambda\}$ are mut. stat. ind. Then by Prop. II.7.1 for any $\{\lambda_1, \ldots, \lambda_n\} \subseteq \Lambda$ $F_{(X_{\lambda_1}, \ldots, X_{\lambda_n})}(x_1, \ldots, x_n) = \prod_{i=1}^{n} F_{X_{\lambda_i}}(x_i)$ which implies that

the cdf of X_{λ_i} is $F_{X_{\lambda_i}}$ and since X_{λ_i} has a discrete distribution it has prob. fn

$p_{X_{\lambda_i}}(x_i) = F_{X_{\lambda_i}}(x_i) - \lim_{\delta_i \downarrow 0} F_{X_{\lambda_i}}(x_i - \delta_i)$.

Then $(X_{\lambda_1}, \ldots, X_{\lambda_n})$ has prob. fn given by

$$p_{(X_{\lambda_1}, \ldots, X_{\lambda_n})}(x_1, \ldots, x_n) = \lim_{\delta_1 \downarrow 0} \cdots \lim_{\delta_n \downarrow 0} P_{(X_{\lambda_1}, \ldots, X_{\lambda_n})}\left(\times_{i=1}^{n} (x_i - \delta_i, x_i]\right)$$

$$= \lim_{\delta_1 \downarrow 0} \cdots \lim_{\delta_n \downarrow 0} \prod_{i=1}^{n} \left(F_{X\pi_i}(a_i) - F_{X\pi_i}(a_i - \delta_i) \right)$$

(recall $P_{(X_1,X_2)}((a_1,b_1] \times (a_2,b_2]) =$
$F_{(X_1,X_2)}(b_1,b_2) - F_{(X_1,X_2)}(a_1,b_2) - F_{(X_1,X_2)}(b_1,a_2) + F_{(X_1,X_2)}(a_1,a_2)$

$= F_{X_1}(b_1)F_{X_2}(b_2) - F_{X_1}(a_1)F_{X_2}(b_2) - F_{X_1}(b_1)F_{X_2}(a_2) + F_{X_1}(a_1)F_{X_2}(a_2)$
$= (F_{X_1}(b_1) - F_{X_1}(a_1))(F_{X_2}(b_2) - F_{X_2}(a_2))$)

$$= \prod_{i=1}^{n} \lim_{\delta_i \downarrow 0} \left(F_{X\pi_i}(a_i) - F_{X\pi_i}(x_i - \delta_i) \right) = \prod_{i=1}^{n} p_{X_i}(a_i)$$

Conversely if $p_{(X\pi_1,\ldots,X\pi_n)}(a_1,\ldots,a_n) = \prod_{i=1}^{n} p_{X\pi_i}(a_i)$ then

$$F_{(X\pi_1,\ldots,X\pi_n)}(a_1,\ldots,a_n) = \sum_{z_1 \le a_1,\ldots,z_n \le a_n} p_{(X\pi_1,\ldots,X\pi_n)}(z_1,\ldots,z_n)$$

$$= \sum_{z_1 \le a_1,\ldots,z_n \le a_n} p_{X\pi_1}(z_1) \cdots p_{X\pi_n}(z_n) = \prod_{i=1}^{n} \sum_{z_i \le a_i} p_{X\pi_i}(z_i)$$

$= \prod_{i=1}^{n} F_{X\pi_i}(a_i)$. So by Prop. IV.7.1 this implies mut. stat. ind.

(ii) The a.c. case is easier!

If $F_{(X\pi_1,\ldots,X\pi_n)}(a_1,\ldots,a_n) = \prod_{i=1}^{n} F_{X\pi_i}(a_i)$ then

$$f_{(X\pi_1,\ldots,X\pi_n)}(a_1,\ldots,a_n) = \frac{\partial^n F_{(X\pi_1,\ldots,X\pi_n)}(a_1,\ldots,a_n)}{\partial a_1 \cdots \partial a_n}$$

$$= \prod_{i=1}^{n} \frac{\partial F_{X\pi_i}(a_i)}{\partial a_i} = \prod_{i=1}^{n} f_{X\pi_i}(a_i).$$

If $f_{(X_{T_1},\ldots,X_{T_n})}(x_1,\ldots,x_n) = \prod_{i=1}^{n} f_{X_{T_i}}(x_i)$ then

$F_{(X_{T_1},\ldots,X_{T_n})}(x_1,\ldots,x_n) = \prod_{i=1}^{n} \int_{-\infty}^{x_i} f_{X_{T_i}}(z_i)\,dz_i$

$= \prod_{i=1}^{n} F_{X_{T_i}}(x_i)$ and the $\{X_T : T \in \Delta\}$

are mut. stat. ind. by Prop. VI.7.1.